§4.2 Richardson's Extrapolation

Suppose $N(h)$ is an algorithm that approximates M and depends on h (step size), and that

$$M = N(h) + K_1 h + K_2 h^2 + K_3 h^3 + \dots \qquad (1)$$

for (possibly unknown) constants $K_1, K_2, \dots$

So

$$|M - N(h)| = O(h).$$

Consider useing a smaller h:

$$M = N(\tfrac{h}{2}) + K_1 \tfrac{h}{2} + K_2 (\tfrac{h}{2})^2 + K_3 (\tfrac{h}{2})^3 + \dots$$

$$\Rightarrow \quad 2M = 2N(\tfrac{h}{2}) + K_1 h + \tfrac{1}{2} K_2 h^2 + \tfrac{1}{4} K_3 h^3 + \dots \qquad (2)$$

Now look at (2) - (1) :

$$M = 2N(\tfrac{h}{2}) - N(h) - \tfrac{1}{2} K_2 h^2 - \tfrac{3}{4} K_3 h^3 - \tfrac{7}{8} K_4 h^4 - \dots \qquad (3)$$

Let $N_2(h) = 2N(\tfrac{h}{2}) - N(h)$. Then

$$|M - N_2(h)| = O(h^2).$$

Next

$$M = N_2(\tfrac{h}{2}) - \tfrac{1}{8} K_2 h^2 - \tfrac{3}{32} K_3 h^3 - \tfrac{7}{64} K_4 h^4 - \dots \qquad (4)$$

So look at $4 \times$ eq 4 - eq 3 :

$$3M = 4N_2(\tfrac{h}{2}) - N_2(h) + \tfrac{3}{8} K_3 h^3 + \tfrac{7}{16} K_4 h^4 + \dots$$

Let $N_3(h) = \tfrac{1}{3}\left(4N_2(\tfrac{h}{2}) - N_2(h)\right)$. Then

$$|M - N_3(h)| = O(h^3).$$

Recapping: if $M = N_1(h) + \sum_{j=1}^{m-1} K_j h^j + O(h^m)$,
then

$$N_2(h) = 2N_1\left(\tfrac{h}{2}\right) - N_1(h) = N_1\left(\tfrac{h}{2}\right) + \left(N_1\left(\tfrac{h}{2}\right) - N_1(h)\right)$$

$$N_3(h) = \tfrac{1}{3}\left(4N_2\left(\tfrac{h}{2}\right) - N_2(h)\right) = N_2\left(\tfrac{h}{2}\right) + \frac{N_2\left(\frac{h}{2}\right) - N_2(h)}{3}$$

$$N_4(h) = N_3\left(\tfrac{h}{2}\right) + \frac{N_3\left(\frac{h}{2}\right) - N_3(h)}{7}$$

$$N_5(h) = N_4\left(\tfrac{h}{2}\right) + \frac{N_4\left(\frac{h}{2}\right) - N_4(h)}{15}$$

$$\vdots$$

$$N_k(h) = N_{k-1}\left(\tfrac{h}{2}\right) + \frac{N_{k-1}\left(\frac{h}{2}\right) - N_{k-1}(h)}{2^{k-1} - 1}$$

$$\Rightarrow \quad M = N_m(h) + O(h^m)$$

Table:

$O(h)$	$O(h^2)$	$O(h^3)$	$O(h^4)$	$O(h^5)$
$N_1(h)$				
$N_1\left(\frac{h}{2}\right)$	$N_2(h)$			
$N_1\left(\frac{h}{4}\right)$	$N_2\left(\frac{h}{2}\right)$	$N_3(h)$		
$N_1\left(\frac{h}{8}\right)$	$N_2\left(\frac{h}{4}\right)$	$N_3\left(\frac{h}{2}\right)$	$N_4(h)$	
$N_1\left(\frac{h}{16}\right)$	$N_2\left(\frac{h}{8}\right)$	$N_3\left(\frac{h}{4}\right)$	$N_4\left(\frac{h}{2}\right)$	$N_5(h)$

Example: Use Richard extrapolation on the Forward difference approximation to find $f'(x_0)$.

Recall: $f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \dots$

$$\Rightarrow \quad f'(x_0) = \frac{f(x)-f(x_0)}{x-x_0} - f''(x_0)\frac{(x-x_0)}{2} - f'''(x_0)\frac{(x-x_0)^2}{3!} - \dots$$

if $x = x_0 + h$, then

$$f'(x_0) = \frac{f(x_0+h)-f(x_0)}{h} - \frac{f''(x_0)}{2}h - \frac{f'''(x_0)}{3!}h^2 - \dots$$

So $N_1(h) = \dfrac{f(x_0+h)-f(x_0)}{h}$, then we can build the table.

Stopping criteria:

Suppose $M = N_1(h) + \sum_{j=1}^{\infty} K_j h^j$ and we want to approximate M accurate to a tolerance τ, then we can build the R.E. table and stop when $|N_{m-1}(h) - N_m(h)| < \tau$. Then $M \approx N_m(h)$.