§4.3 Elements of Numerical Integration

Recall: Lagrange Polynomials:

$$f(x) = P_n(x) = \sum_{k=0}^{n} f(x_k) L_k(x) + R_n(x)$$

where
$$R_n(x) = \prod_{k=0}^{n} (x - x_k) \frac{f^{(n+1)}(\xi(x))}{(n+1)!},$$

$a \leq x_0 < x_1 < x_2 < \cdots < x_n \leq b$ and $\xi(x) \in [a, b]$.

So
$$\int_a^b f(x)\,dx = \int_a^b \sum_{k=0}^{n} f(x_k) L_k(x)\,dx$$
$$+ \int_a^b \prod_{k=0}^{n} (x - x_k) \frac{f^{(n+1)}(\xi(x))}{(n+1)!}\,dx$$

The question becomes: How do we pick the x_k's for various quadrature methods.

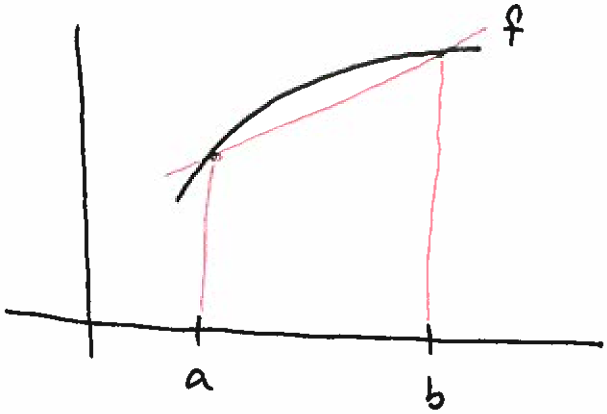
Common Endpoint Quadratures:

Here we have $a = x_0 < x_1 < \cdots < x_n = b$

$n = 1$: Trapezoidal Rule ($h = b - a$)

$$\int_a^b f(x)\,dx = \frac{h}{2}[f(a) + f(b)] - \frac{h^3}{12} f''(\xi)$$

where $a < \xi < b$.

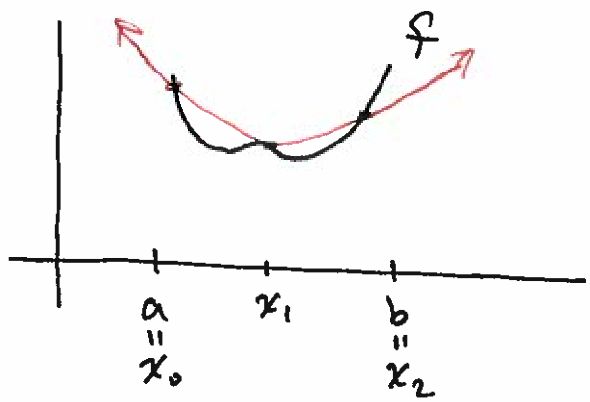


Area under the curve is approximated by a trapezoid.

$n=2$: Simpson's Rule $\left(h=\frac{b-a}{2},\ x_k=a+kh\right)$

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right] - \frac{h^5}{90}f^{(4)}(\xi)$$

where $a<\xi<b$.



Area under the curve is approx. by a quadratic.

$n=3$: Simpson's 3/8 Rule $\left(h=\frac{b-a}{3},\ x_k=a+kh\right)$

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\right] - \frac{3h^5}{80}f^{(4)}(\xi),$$

where $a<\xi<b$.

Fit a cubic through four points.

$n=4$: $\left(h=\frac{b-a}{4},\ x_k=a+kh\right)$

$$\int_a^b f(x)\,dx = \frac{2h}{45}\left(7f(x_0)+32f(x_1)+12f(x_2) + 32f(x_3)+7f(x_4)\right] - \frac{8h^7}{945}f^{(6)}(\xi)$$

where $a<\xi<b$.

Degree of Accuracy or Precision

If a quadrature is exact for $f(x)=x^k$, $k=1,2,\dots,n$, then we say the quadrature has (or is) precision n (or n^{th} degree of accuracy).

Example Simpson's Rule

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)$$

If $f(x)=x^k$ then the LHS is

$$\frac{b^{k+1}-a^{k+1}}{k+1}$$

The RHS is

$$\frac{b-a}{6}\left(a^k + 4\left(\frac{a+b}{2}\right)^k + b^k\right)$$

k	LHS	RHS
0	$b-a$	$\frac{b-a}{6}(1+4+1) = b-a$
1	$\frac{b^2-a^2}{2}$	$\frac{b-a}{6}\left(a + 2(a+b)^1 + b\right) = \frac{3(a+b)(b-a)}{6}$ $= \frac{b^2-a^2}{2}$
2	$\frac{b^3-a^3}{3}$	$\frac{b-a}{6}\left(a^2 + 4\left(\frac{a+b}{2}\right)^2 + b^2\right)$ $= \frac{b-a}{6}\left(a^2 + (a^2+2ab+b^2) + b^2\right)$ $= \frac{b-a}{6} \cdot 2(a^2+ab+b^2) = \frac{b^3-a^3}{3}$
3	$\frac{b^4-a^4}{4}$	$\frac{b-a}{6}\left(a^3 + 4\left(\frac{a+b}{2}\right)^3 + b^3\right) \overset{CAS}{=} \frac{b^4-a^4}{4}$
4	$\frac{b^5-a^5}{5}$	$\frac{b-a}{6}\left(a^4 + 4\left(\frac{a+b}{2}\right)^4 + b^4\right) \overset{CAS}{\neq} \frac{b^5-a^5}{5}$

So Simpson's Rule has precision 3.

In general, if n is even, then the degree of precision is $n+1$ and if n is odd, the degree of precision is n.

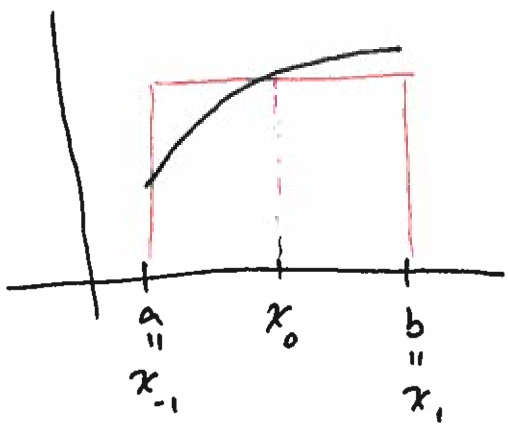
Open Newton-Cotes Formulas

Here we need not use endpoints:

$n = 0$: Midpoint Rule ($x_0 = \frac{a+b}{2}$, $h = \frac{b-a}{2}$)

$$\int_a^b f(x)\,dx = 2h\,f(x_0) + \frac{h^3}{3} f''(\xi)$$

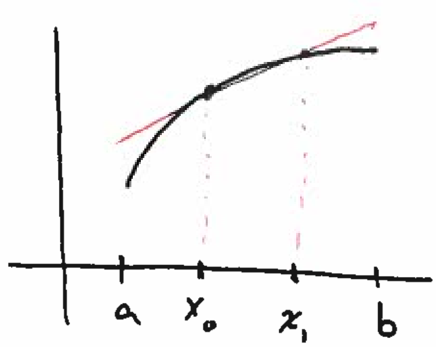
where $a < \xi < b$.



$n = 1$: ($h = \frac{b-a}{3}$, $\cancel{x_k = a + kh}$ $x_k = a + (k+1)h$)

$$\int_a^b f(x)\,dx = \frac{3h}{2}\left(f(x_0) + f(x_1)\right) + \frac{3h^3}{4} f''(\xi),$$

where $a < \xi < b$



See page 199 for $n=2, n=3$

<u>Example</u> Find the degree of precision for the Midpoint Rule.

$$\int_a^b f(x)\,dx \approx (b-a)\, f\left(\frac{a+b}{2}\right)$$

for $f(x) = x^K$, the RHS is $\dfrac{b^{K+1} - a^{K+1}}{K+1}$ and the LHS is $(b-a)\left(\frac{a+b}{2}\right)^K$.

K	RHS	LHS
0	$b-a$	$b-a$
1	$\frac{b^2-a^2}{2}$	$(b-a)\left(\frac{a+b}{2}\right) = \frac{b^2-a^2}{2}$
2	$\frac{b^3-a^3}{3}$	$(b-a)\left(\frac{a+b}{2}\right)^2 \neq \frac{b^3-a^3}{3}$

So precision 1