§4.4 Composite Numerical Integration

Recall from calculus I that

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Composite numerical integration uses this.
Consider the interval $a \le x \le b$. We can split (partition) the interval into ~~subintegra~~ subintervals:
let $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, then

$$\int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \dots + \int_{x_{n-1}}^{x_n} f(x)dx.$$

On each integral on the right, we could apply one of the rules from the previous section.

Composite Simpson's Method

Let n be a positive even integer, $h = \frac{b-a}{n}$, $x_k = a + kh$. Then

$$\int_a^b f(x)dx = \sum_{j=1}^{n/2} \int_{x_{2j-2}}^{x_{2j}} f(x)dx$$

$$= \sum_{j=1}^{n/2} \left(\frac{h}{3}\left(f(x_{2j-2}) + 4f(x_{2j-1}) + f(x_{2j})\right) - \frac{h^5}{90} f^{(4)}(\xi_j) \right)$$

where $\xi_j \in [x_{2j-2}, x_{2j}]$ and provided $f \in C^4[a,b]$.

$$\int_a^b f(x)\,dx = \frac{h}{3}\left(f(x_0) + 2\sum_{j=1}^{\frac{n}{2}-1} f(x_{2j}) + 4\sum_{j=1}^{\frac{n}{2}} f(x_{2j-1}) + f(b)\right) - \frac{h^5}{90}\sum_{j=1}^{n/2} f^{(4)}(\xi_j)$$

The error $E_{S_n}(f) = -\frac{h^5}{90}\sum_{j=1}^{n/2} f^{(4)}(\xi_j)$.

The Extreme Value Theorem gives $f \in C^4[a,b] \Rightarrow$ $\exists\, m, M \ni \; m \le f^{(4)}(x) \le M$, $\forall x \in [a,b]$. Thus $\exists\, B > 0 \ni \; |f^{(4)}(\xi_j)| \le B$, for each j and

$$|E_{S_n}(f)| \le \frac{h^5}{90}\cdot\frac{n}{2}\,B = \frac{B(b-a)^5}{180\,n^4}.$$

You may remember this from calculus II.

<u>Interval doubling and stopping criteria</u>

Consider the sequence $\{S_n\}$ where

$$S_0 = \int_a^b f(x)\,dx$$

$$S_1 = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx \quad \text{where } h = \frac{b-a}{2},\; x_k = a + kh$$

$$S_2 = \sum_{j=1}^{2^2} \int_{x_{j-1}}^{x_j} f(x)\,dx \quad \text{where } h = \frac{b-a}{4},\; x_k = a + kh$$

$$\vdots$$

$$S_m = \sum_{j=1}^{2^m} \int_{x_{j-1}}^{x_j} f(x)\,dx \quad \text{where } h = \frac{b-a}{2^m}$$

$$x_k = a + kh$$

and each integral uses the same quadrature rule. Then as $m \to \infty$ $E(f) \to 0$ and $\{S_m\}$ is a convergent sequence. We generally use

$$|S_{m-1} - S_m| < \text{Tolarance}$$

as a stopping criteria.

Note for Composite Simpson's Method

$$S_0 = \frac{(b-a)}{6}\left(f(a) + 4f(x_1) + f(b)\right)$$

$$S_1 = \frac{(b-a)}{12}\left(f(a) + 4f(\tilde{x}_1) + 2f(\hat{x}_1) + 4f(\tilde{x}_2) + f(b)\right)$$

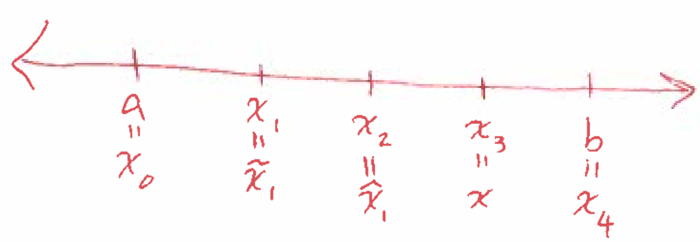
$$\cancel{= \frac{1}{2} S_0 + \frac{4h}{3}(f f(}$$

$$\cancel{= \frac{1}{2}\text{bob} + \frac{4h}{3}\text{sally} + \frac{2h}{3}}$$

$$= \frac{h}{3}\left(\text{bob} + 4\,\text{sally} + 2\,\text{joe}\right)$$

where $\text{bob} = f(a) + f(b)$, $h = \frac{b-a}{4}$,

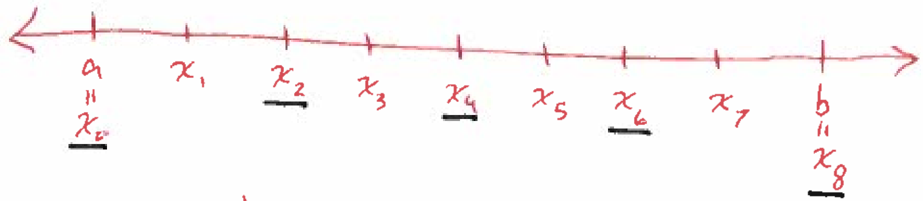
$\text{Sally} = \sum_{j=1}^{2} f(x_{2j-1})$, $\text{joe} = f(x_2)$



for S_2, $h = \frac{(b-a)}{8}$, $x_k = a + kh$

let doug = sally + joe and

fred $= \sum_{j=1}^{4} f(x_{2j-1})$



— old nodes used in S_1

then

$$S_2 = \frac{h}{3}(\text{bob} + 4\,\text{fred} + 2\,\text{doug})$$

⋮

In general:

Inputs: a, b, tol, f, m ← max # of doublings

Outputs: approximation for $\int_a^b f(x)\,dx$ or error message.

Set
$S1 = f(a) + f(b);$
$S2 = f(a+h);$
$S3 = 0;$
$h = (b-a)/2;$ $n = 1$
notdone = true ← approx1 = h*(~~bob~~ S1 + 4*S2)/3;

loop while notdone

$h = h/2$

$S3 = S3 + S2$

$S2 = f(a+h) + f(a+3h) + \dots + f(b-h)$

approx2 = h*(S1 + 4*S2 + 2S3)/3;

If |approx1 - approx2| < Tol

then ~~approx1 = approx2;~~

notdone = True;

```
            Elseif n > m
              then output error message;
                   notdone = True;
              Else n=n+1;
                   approx1 = approx2;
          end (if)
      end (loop)
      ~~Return~~ Output approx1
end (Rutine)
```

Note in Matlab: If $h = (b-a)/n$, then

$x = [a^{+h} : 2h : b];$

gives a vector of (a+h, a+3h, a+5h, ..., b-h)

$\boxed{\text{Program this in Matlab}}$

Compsite Trapezoidal Method

$$S_0 = \frac{h}{2}\left(f(x_0) + f(x_1)\right), \quad h = b-a, \quad x_k = a + kh$$

$$S_1 = \frac{h}{2}\left(f(x_0) + 2f(x_1) + f(x_2)\right), \quad h = \frac{b-a}{2}, \quad x_k = a + kh$$

$$S_2 = \frac{h}{2}\left(f(x_0) + f(x_n) + 2\sum_{j=1}^{3} f(x_j)\right), \quad h = \frac{b-a}{4}$$

$$\vdots$$

$$S_n = \frac{h}{2}\left(f(x_0) + f(x_n) + 2\sum_{j=1}^{2^n-1} f(x_j)\right), \quad h = \frac{b-a}{2^n}$$

Note we can rewrite this as

$$S_n = h\left(\frac{f(a)+f(b)}{2} + \sum_{j=1}^{2^n-1} f(x_j)\right)$$

In general,

If $h = \frac{(b-a)}{n}$, $x_k = a + kh$, then

$$\int_a^b f(x)\,dx = h\left(\frac{f(x_0)+f(x_n)}{2} + \sum_{j=1}^{n-1} f(x_j)\right) - \frac{h^3}{12}\sum_{j=1}^{n} f''(\xi_j)$$

where $x_{j-1} \le \xi_j \le x_j$

If $|f''(x)| \le B$ for $a \le x \le b$, then

$$|E_{T_n}(f)| \le \frac{B(b-a)^3}{12n^2}.$$

You may remember this from calculus II

Programing composite trapezoidal Method with Interval doubling

Inputs: a, b, f, Tol, m
Outputs: approximation to $\int_a^b f(x)dx$ or error message

```
Set   h = b - a ;
      n = 1 ;
      S1 = (f(a) + f(b))/2 ;
      notdone = True ;
loop While notdone
      h = h/2
      n = n+1
      S2 = S1/2 + h(f(a+h) + f(a+3h) + f(a+5h) + ... + f(b-h))
      If (|S1 - S2| > Tol and n < m)
            S1 = S2 ;
         Else
            notdone = False ;
      end (if)
end (loop)
If n ≥ m
   Output a error message
Output S2
end (Rutine)
```

Look at this in Matlab.