§4.5 Romberg Integration

Romberg Integration applies Richardson Extrapolation to the Composite Trapezoidal Method.

Recall:

If $M = N_1(h) + k_1 h + k_2 h^2 + \dots$, then we define

$$N_j(h) = N_{j-1}\left(\frac{h}{2}\right) + \frac{N_{j-1}\left(\frac{h}{2}\right) - N_{j-1}(h)}{2^{j-1} - 1}, \quad j = 2,3,4,\dots$$

and then $M = N_j(h) + O(h^j)$

In a similar ~~fashio~~ way, if

$$M = N_1(h) + k_1 h^2 + k_2 h^4 + k_3 h^6 + \dots$$

then define

$$N_j(h) = N_{j-1}\left(\frac{h}{2}\right) + \frac{N_{j-1}\left(\frac{h}{2}\right) - N_{j-1}(h)}{4^{j-1} - 1}, \quad j = 2,3,4,\dots$$

(see page 186) and then

$$M = N_j(h) + O(h^{2j})$$

Let $T_n(f) = h_n\left(\frac{f(a) + f(b)}{2} + \sum_{j=1}^{n-1} f(x_j)\right)$

where $h_n = \frac{b-a}{n}$, $x_K = a + K h_n$. Then

$$M = \int_a^b f(x)\,dx = T_n(f) + k_1 h_n^2 + k_2 h_n^4 + k_3 h_n^6 + \dots$$

Thus we can apply the page 186 version of Richardson Extrapolation.

Let $R_{1,1} = \frac{h_1}{2}[f(a) + f(b)]$

$$R_{k,1} = \frac{1}{2} R_{k-1,1} + h_{2^{k-1}} \sum_{j=1}^{2^{k-2}} f(a + (2j-1)h_{2^{k-1}})$$

and $$R_{k,l} = R_{k,l-1} + \frac{R_{k,l-1} - R_{k-1,l-1}}{4^{l-1} - 1}$$

Look at code posted on web.

Example Use Romberg integration to approximate $\int_0^5 x^2 e^{-x^2} dx$ accurate to 10^{-4}.

Change myf.m to x.^2.*exp(-x.*x)

Run romberg code in Matlab with maxit = 15

Romberg table:

$R_{1,1}$				
$R_{2,1}$	$R_{2,2}$			
$R_{3,1}$	$R_{3,2}$	$R_{3,3}$		
$R_{4,1}$	$R_{4,2}$	$R_{4,3}$	$R_{4,4}$	
$R_{5,1}$	$R_{5,2}$	$R_{5,3}$	$R_{5,4}$	$R_{5,5}$

Note that $R_{n,*}$ only depends on $R_{n-1,*}$. We only need to keep track of the current row and the previous row. Stopping when $|R_{n,n} - R_{n-1,n-1}| < \text{tolerance}$.