§4.6 Adaptive Quadrature Methods

Recall Simpson's Rule:

$$S(a,b) = \frac{h_1}{3}\left(f(a) + 4f\left(\tfrac{a+b}{2}\right) + f(b)\right)$$

where $h_1 = \frac{b-a}{2}$

Then $$\int_a^b f(x)\,dx = S(a,b) - \frac{h_1^5}{90} f^{(4)}(\xi)$$

for some $\xi \in (a,b)$. Then composite Simpson's Method gives

$$\int_a^b f(x)\,dx = S\left(a, \tfrac{a+b}{2}\right) + S\left(\tfrac{a+b}{2}, b\right) - \left(\frac{h_1}{2}\right)^4 \frac{(b-a)}{180} f^{(4)}(\xi)$$

for some $\xi \in (a,b)$.

Combining these two we get that

$$\frac{h_1^5}{90} f^{(4)}(\xi) \approx \frac{16}{15}\left(S(a,b) - S\left(a, \tfrac{b+a}{2}\right) - S\left(\tfrac{a+b}{2}, b\right)\right).$$

So

$$\left|\int_a^b f(x)\,dx - S\left(a, \tfrac{a+b}{2}\right) - S\left(\tfrac{a+b}{2}, b\right)\right| \approx$$

$$\left|\frac{1}{16}\left(\frac{h_1^5}{90}\right) f^{(4)}(\xi)\right| \approx \frac{1}{15}\left|S(a,b) - S\left(a, \tfrac{a+b}{2}\right) - S\left(\tfrac{a+b}{2}, b\right)\right|.$$

So, if we wish to approximate $\int_a^b f(x)\,dx$ accurate to a tolerance of ε, then

Calculate $S_1 = S(a,b)$

$S_2 = S(a, \frac{a+b}{2})$

$S_3 = S(\frac{a+b}{2}, b)$

If $|S_1 - S_2 - S_3| < 15\varepsilon$

then $S_2 + S_3$ is the approximation

Else apply the method on $\int_a^{\frac{a+b}{2}} f(x)\,dx$ and $\int_{\frac{a+b}{2}}^b f(x)\,dx$ each with tolerance $\frac{\varepsilon}{2}$.

Look at code aq.m on web site and Algorithm 4.3 on page 224.