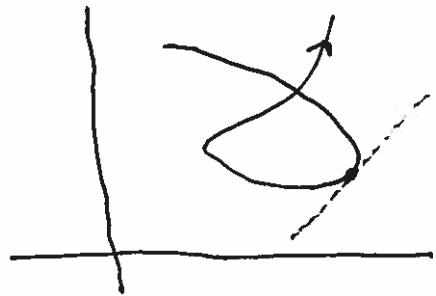


## §10.2 Calculus with Parametric Curves



Tangent lines:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Example (i) Let  $x = 2\cos(t)$ ,  $y = 3\sin(t)$  and find the equation of the tangent line to the curve when  $t = \frac{\pi}{6}$

$$\frac{dy}{dt} = 3\cos(t) \Big|_{\frac{\pi}{6}} = 3 \cdot \frac{\sqrt{3}}{2}$$

$$\frac{dx}{dt} = -2\sin(t) \Big|_{\frac{\pi}{6}} = -2 \cdot \frac{1}{2} = -1$$

so  $\frac{dy}{dx} = \frac{3\cos(t)}{-2\sin(t)} \Big|_{\frac{\pi}{6}} = -\frac{3\sqrt{3}}{2}$

Note that  $x(\frac{\pi}{6}) = \sqrt{3}$  and  $y(\frac{\pi}{6}) = \frac{3}{2}$

so the tangent line is

$$y = \frac{3}{2} - \frac{3\sqrt{3}}{2}(x - \sqrt{3})$$

For concavity we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left[ \frac{dy}{dx} \right] = \frac{\frac{d}{dt} \left[ \frac{dy}{dx} \right]}{\frac{dx}{dt}}$$

Example(ii) Find the concavity for  $x=2\cos(t)$ ,  $y=3\sin(t)$  at  $t=\frac{\pi}{6}$ .

From example(i):

$$\frac{dy}{dx} = -\frac{3}{2} \cot(t)$$

so  $\frac{d}{dt} \left[ \frac{dy}{dx} \right] = \frac{3}{2} \csc^2(t) \Big|_{\pi/6} = 6$

and we still have

$$\frac{dx}{dt} = -2\sin(t) \Big|_{\pi/6} = -1$$

thus

$$\frac{d^2y}{dx^2} = \frac{\frac{3}{2} \csc^2(t)}{-2\sin(t)} = -\frac{3}{4} \csc^3(t) \Big|_{\pi/6} = -6$$

We could continue in the same way to find

$$\frac{d^3y}{dx^3} = \frac{\frac{d}{dt} \left[ \frac{d^2y}{dx^2} \right]}{\frac{dx}{dt}}$$

and higher derivatives.

## Arc Length in Parametric

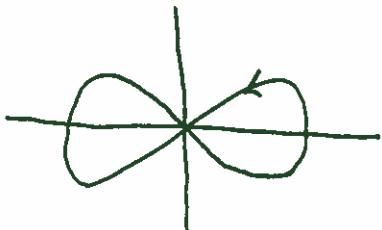
Recall:

$$ds = \sqrt{dx^2 + dy^2}$$

so

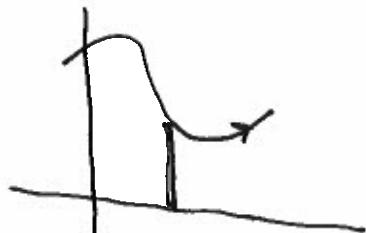
$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Example(iii) Let  $x = \cos(2t)$ ,  $y = \sin(4t)$ ,  $0 \leq t \leq 2\pi$ .  
 Find the length of the curve.



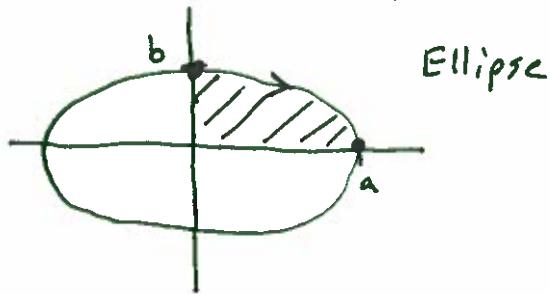
$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(-2\sin(2t))^2 + (4\cos(4t))^2} dt \\ &= 18.8589 \end{aligned}$$

## Area in Parametric



$$dA = y dx \text{ or } x dy$$

Example Find the area enclosed in  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .  
for  $a > b > 0$



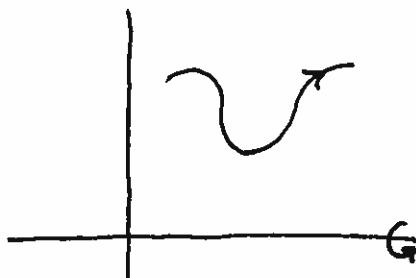
Elliptic

Parametric

$$\begin{aligned} x &= a \sin(t) \\ y &= b \cos(t) \\ 0 \leq t \leq \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} A &= 4 \int_0^{\pi/2} (a \sin(t)) (b \cos(t)) dt \\ &= +4ab \int_0^{\pi/2} \frac{1 + \cos(2t)}{2} dt \\ &= +2ab \left( t + \frac{1}{2} \sin(2t) \right) \Big|_0^{\pi/2} \\ &= ab\pi \end{aligned}$$

Surface area of Revolution, Volume of Revolution, ...



$$dS = 2\pi R ds$$

$$dV = \pi R^2 dx$$

where  $R = y$  in the picture

So all of the old formulas can be converted to parametric by substituting in the parametric equations for  $x$  and  $y$ .