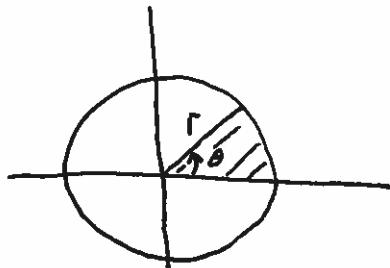


§10.4 Area and Length in Polar

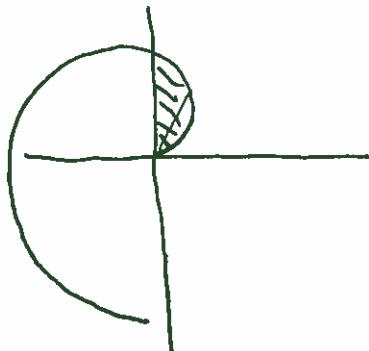
Area in Polar

Circular sector:

$$A = \frac{1}{2} r^2 \theta$$



Example (i) $r = \theta$, $0 \leq \theta \leq \frac{\pi}{2}$



$$dA = \frac{1}{2} \theta^2 d\theta$$

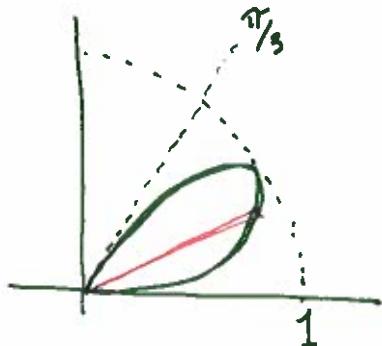
$$A = \frac{1}{2} \int_0^{\pi/2} \theta^2 d\theta$$

$$= \frac{1}{6} \theta^3 \Big|_0^{\pi/2}$$

$$= \frac{1}{6} \left(\frac{\pi}{2}\right)^3$$

$$= \frac{\pi^3}{56}$$

Example (ii) One loop of $r = \sin(3\theta)$



Tangents at the poles:

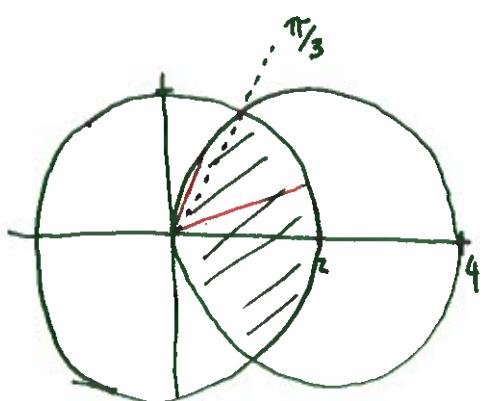
$$r=0 \Rightarrow 3\theta = 0, \pi, 2\pi, \dots$$

$$\theta = 0, \frac{\pi}{3}$$

$$A = \frac{1}{2} \int_0^{\pi/3} \sin^2(3\theta) d\theta$$

$$\begin{aligned}
 &= \frac{1}{4} \int_0^{\pi/3} (1 - \cos(6\theta)) d\theta \\
 &= \frac{1}{4} \left(\theta - \frac{1}{6} \sin(6\theta) \right) \Big|_0^{\pi/3} \\
 &= \frac{\pi}{12}
 \end{aligned}$$

Example (iii) Inside $r=2$ and $r=4\cos(\theta)$



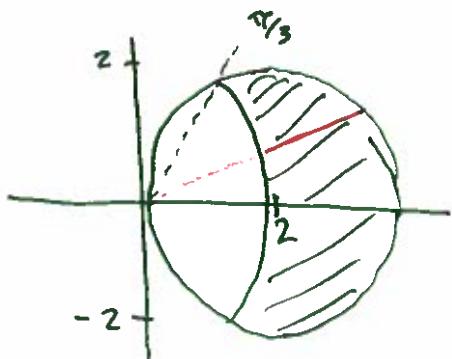
Intersection points:

$$\begin{aligned}
 2 &= 4\cos(\theta) \\
 \cos(\theta) &= \frac{1}{2} \\
 \theta &= \pm \frac{\pi}{3}
 \end{aligned}$$

We have symmetry about x -axis,
but the region is not θ -simple. Thus we need
two integrals to calculate the area.

$$\begin{aligned}
 \frac{1}{2}A &= \frac{1}{2} \int_0^{\pi/3} 2^2 d\theta + \frac{1}{2} \int_{\pi/3}^{\pi/2} (4\cos(\theta))^2 d\theta \\
 \Rightarrow A &= 4 \int_0^{\pi/3} d\theta + 16 \int_{\pi/3}^{\pi/2} \cos^2(\theta) d\theta \\
 &\quad \blacksquare \\
 &= \frac{8\pi}{3} - 2\sqrt{3}
 \end{aligned}$$

Example (iv) Inside $r=4\cos(\theta)$ and outside $r=2$



Here we have an outside-inside areas case:

$$\begin{aligned}
 A &= \int_0^{\pi/3} ((4\cos(\theta))^2 - 2^2) d\theta \\
 &\quad \blacksquare \\
 &= \frac{4\pi}{3} + 2\sqrt{3}
 \end{aligned}$$

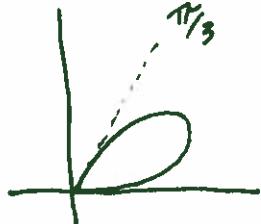
Length of curves

Use parametric: $r = f(\theta) \Rightarrow x = f(\theta)\cos(\theta)$,
 $y = f(\theta)\sin(\theta)$ to get

$$\begin{aligned}x'^2 + y'^2 &= (f'(\theta)\cos(\theta) - f(\theta)\sin(\theta))^2 \\&\quad + (f'(\theta)\sin(\theta) + f(\theta)\cos(\theta))^2 \\&= f'^2(\cos^2(\theta) + \sin^2(\theta)) \\&\quad + f^2(\sin^2(\theta) + \cos^2(\theta)) \\&= f'^2 + f^2\end{aligned}$$

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Example (v) One loop of $r = \sin(3\theta)$



From example (ii) we get the tangents at the poles are $\theta = 0, \pi/3$

so,

$$S = \int_0^{\pi/3} \sqrt{(\sin(3\theta))^2 + (3\cos(3\theta))^2} d\theta$$



$$= 2.22748$$

Note that this integral is difficult to do by hand.