

§10.1 Sequences

Def A sequence is a function whose domain is a subset of integers.

Examples

(i) $\{1, 1, 2, 3, 5, 8, 13, \dots\}$ Fibonacci sequence

(ii) $a_n = \frac{n}{n+1} \Leftrightarrow \left\{ \frac{n}{n+1} \right\}_{n=0}^{\infty} \Leftrightarrow \left\{ 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots \right\}$

(iii) $\{1, 0, -1, 0, 1, 0, -1, 0, \dots\} \Leftrightarrow b_k = \cos\left(\frac{k\pi}{2}\right), k \geq 0$

(iv) $f_0 = 1, f_n = n \cdot f_{n-1} \text{ for } n=1, 2, 3, \dots$

$$\text{Then } f_1 = (1)(f_0) = 1$$

$$f_2 = (2)(f_1) = 2$$

$$f_3 = (3)(f_2) = (3)(2)(1) = 6$$

$$f_4 = (4)(f_3) = (4)(3)(2)(1) = 24$$

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$$f_k = k! \quad (\text{k factorial})$$

Convergence

Def $\lim_{n \rightarrow \infty} a_n = L$ if

$\forall \varepsilon > 0, \exists N \in \mathbb{Z}^+ \ni n > N \Rightarrow |a_n - L| < \varepsilon$

and we write $a_n \rightarrow L$ (as $n \rightarrow \infty$).

In the previous example we can use the Squeeze theorem

The Squeeze theorem

If f is ~~continuous~~ defined on an interval I , containing a , and $h(x) \leq f(x) \leq g(x)$ for x in I , and

$$\lim_{x \rightarrow a} h(x) = L = \lim_{x \rightarrow a} g(x)$$

then

$$\lim_{x \rightarrow a} f(x) = L \text{ also.}$$

So in the example

$$-1 \leq \sin(n) \leq 1, \quad \forall n \in \mathbb{Z}^+$$

$$\Rightarrow -\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$$

and $\lim_{n \rightarrow \infty} -\frac{1}{n} = 0 = \lim_{n \rightarrow \infty} \frac{1}{n}$

$$\therefore \lim_{n \rightarrow \infty} \frac{\sin(n)}{n} = 0$$

Example (viii) $c_n = 1 + (-1)^n$

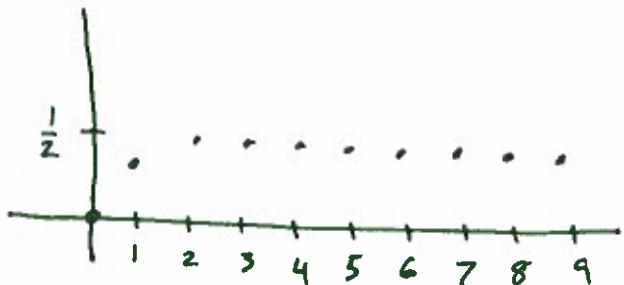
graph it.



Since $c_{2n} = 2$ and $c_{2n+1} = 0 \quad \forall n \in \mathbb{Z}^+$

$\lim_{n \rightarrow \infty} c_n = \text{DNE}$ and we say $\{c_n\}$ diverges.

Example (vi) $a_n = n^2 \left(1 - \cos\left(\frac{1}{n}\right)\right)$
graph using calculator and Mathematica.



$$\lim_{n \rightarrow \infty} a_n = \frac{1}{2}$$

Example (vii) $b_k = \frac{\sin(k)}{k}$

graph it

$$\lim_{k \rightarrow \infty} \frac{\sin(k)}{k} = 0$$

Theorem If f is a function defined on \mathbb{R} and $a_n = f(n)$ for $n = 1, 2, 3, \dots$, and $\lim_{x \rightarrow \infty} f(x) = L$, then $\lim_{n \rightarrow \infty} a_n = L$.

Because of this theorem, all the properties of limits still hold for sequences. See page 693.

Theorem f continuous at L and $a_n \rightarrow L$

$$\Rightarrow \lim_{n \rightarrow \infty} f(a_n) = f(L)$$

Example (ix) $d_n = \tan^{-1}\left(\frac{n}{n+1}\right)$

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \Rightarrow \lim_{n \rightarrow \infty} d_n = \tan^{-1}(1) = \frac{\pi}{4}$$

Example (x) $\{ar^n\}_{n=0}^{\infty}$

This is called a geometric sequence. Other ways of defining it are

$$\{a, ar, ar^2, ar^3, \dots\}$$

or recursively as $c_0 = a$ and $c_{n+1} = rc_n$ for $n=0, 1, 2, 3, \dots$

$$\lim_{n \rightarrow \infty} ar^n = \begin{cases} 0 & \text{if } |r| < 1 \\ \text{DNE} & \text{if } |r| > 1 \text{ or } r = -1 \\ a & \text{if } r = 1 \end{cases}$$

Def Let $\{a_n\}$ be a sequence of real numbers.

- If $a_{n+1} > a_n \ \forall n$, then $\{a_n\}$ is increasing.
- If $a_{n+1} < a_n \ \forall n$, then $\{a_n\}$ is decreasing.
- If $\exists M \ni |a_n| \leq M \ \forall n$, then $\{a_n\}$ is bounded.

We can also define "nondecreasing", "nonincreasing", "bounded above" and "bounded below" in similar ways.

Def $\{a_n\}$ is a Monotonic sequence of real numbers if it is either nondecreasing or nonincreasing.

Theorem (MCT)

If $\{a_n\}$ is a bounded, Monotonic sequence of real numbers, then it converges.

Example (xi)

Let $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$.

(a) Write and approximate the values of a_2, a_3, a_4 and a_5 .

(b) Show that $\{a_n\}$ is bounded above by 3.

(c) Show that $\{a_n\}$ is an increasing sequences

(d) Conclude that $\{a_n\}$ converges and find its limit.

$$(a) a_2 = \sqrt{2 + a_1} = \sqrt{2 + \sqrt{2}} \approx 1.84776$$

$$a_3 = \sqrt{2 + a_2} = \sqrt{2 + \sqrt{2 + \sqrt{2}}} \approx 1.96157$$

$$a_4 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}} \approx 1.99037$$

$$a_5 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}} \approx \cancel{1.99037} 1.99759$$

(b) Proof by Induction:

(B.C.) Show $a_1 \leq 3$:

$a_1 = \sqrt{2} \leq 3 \Leftrightarrow 2 \leq 9$ and \sqrt{x} is increasing.

(I.H.) Assume that $a_k \leq 3$ for some $k \in \mathbb{Z}^+$.

(I.S.) Show that $(\text{I.H.}) \Rightarrow a_{k+1} \leq 3$.

Look at $a_{k+1} = \sqrt{2+a_k}$

$$\Leftrightarrow a_{k+1}^2 = 2 + a_k$$

then by (I.H.), $a_{k+1}^2 \leq 2 + 3 = 5$

$$\Rightarrow a_{k+1}^2 \leq 9$$

$$\Leftrightarrow a_{k+1} \leq 3$$

\therefore By mathematical induction, $a_n \leq 3 \forall n \in \mathbb{Z}^+$.

(c) Proof by Induction

(B.C.) Show $a_1 < a_2$.

$$a_1 = \sqrt{2} < \sqrt{2+\sqrt{2}} = a_2 \quad \checkmark$$

(I.H.) Assume that $a_k < a_{k+1}$ for some $k \in \mathbb{Z}^+$

(I.S.) Show that $(\text{I.H.}) \Rightarrow a_{k+1} \leq a_{k+2}$.

Look at $a_k < a_{k+1}$ (by I.H.)

$$\Rightarrow 2 + a_k < 2 + a_{k+1}$$

$$\Rightarrow \sqrt{2+a_k} < \sqrt{2+a_{k+1}} \quad \text{b/c } \sqrt{x} \text{ is increasing}$$

$$\Rightarrow a_{n+1} < a_{n+2} \text{ by recursive def.}$$

$\therefore a_n < a_{n+1} \forall n \in \mathbb{Z}^+$ and $\{a_n\}$ is an increasing sequence of real numbers.

(c) Since $\{a_n\}$ is a bounded above, increasing sequence, it converges by the MCT.

Let

$$a = \lim_{n \rightarrow \infty} a_n, \text{ then}$$

$$a_{n+1} = \sqrt{2 + a_n} \xrightarrow{n \rightarrow \infty}$$

$$a = \sqrt{2+a} \Rightarrow$$

$$a^2 = 2 + a \Rightarrow$$

$$a^2 - a - 2 = 0 \Rightarrow$$

$$(a-2)(a+1) = 0 \Rightarrow$$

$$a = 2 \text{ or } a = -1.$$

$a = -1$ is extraneous, so

$$\lim_{n \rightarrow \infty} a_n = 2$$

In the sections that follow, we will be talking a lot about sequences and series.

Convergence of series is based on convergence of sequences. Work enough problems so that you are comfortable with this concept.