

## §11.2 Series

Example (i)  $s = 0.9999\dots$  Write as an infinite series.

$$\begin{aligned} s &= \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \dots \\ &= \sum_{k=1}^{\infty} 9\left(\frac{1}{10}\right)^k \end{aligned}$$

This is an example of a "geometric series".

Let  $\{a_n\}$  be an infinite sequence of real numbers.

Let  $\{s_n\}$  be the sequence of partial sums generated by

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$s_n = \sum_{k=1}^n a_k$$

If  $\{s_n\}$  converges to a number  $s$ , then we say the series  $\sum_{k=1}^{\infty} a_k$  converges and we write

$$s = \sum_{k=1}^{\infty} a_k$$

If  $\lim_{n \rightarrow \infty} s_n = \text{DNE}$ , then we say  $\sum_{k=1}^{\infty} a_k$  diverges

## Geometric Series

$$\sum_{k=0}^{\infty} ar^k$$

The sequence of partial sums are

$$S_1 = a$$

$$S_2 = a + ar$$

$$S_3 = a + ar + ar^2$$

$$\text{But then: } S_n = a + ar + ar^2 + \dots + ar^{n-1}$$

$$rS_n = \underline{ar + ar^2 + \dots + ar^{n-1} + ar^n}$$

$$\text{Subtract: } S_n - rS_n = a - ar^n$$

$$\text{or: } S_n = \frac{a(1-r^n)}{1-r} = \sum_{k=0}^{n-1} ar^k$$

Now let  $n \rightarrow \infty$  to get

$$\sum_{k=0}^{\infty} ar^k = \begin{cases} \frac{a}{1-r}, & |r| < 1 \\ \text{diverges}, & |r| \geq 1 \end{cases}$$

Example (ii) show that  $0.999\dots = 1$

$$0.999\dots = \sum_{k=1}^{\infty} 9\left(\frac{1}{10}\right)^k = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \dots$$

First term =  $\frac{9}{10}$ , common ratio =  $\frac{1}{10}$

That is  $a = \frac{9}{10}$ ,  $r = \frac{1}{10}$ . Note  $|r| < 1$ , so

$$0.999\dots = \frac{a}{1-r} = \frac{\frac{9}{10}}{1-\frac{1}{10}} = 1$$

Example (iii)) write  $0.219219219\dots$  as a fraction

$$\begin{aligned} .219219219\dots &= \frac{219}{1000} + \frac{219}{1000^2} + \frac{219}{1000^3} + \dots \\ &= \frac{\frac{219}{1000}}{1 - \frac{1}{1000}} \\ &= \frac{219}{999} \end{aligned}$$

### Telescoping Series

Such as  $\sum_{k=0}^{\infty} (a_k - a_{k+1})$ . The partial sums

are

$$S_1 = a_0 - a_1$$

$$S_2 = (a_0 - a_1) + (a_1 - a_2) = a_0 - a_2$$

$$S_3 = (a_0 - a_1) + (a_1 - a_2) + (a_2 - a_3) = a_0 - a_3$$

$$S_n = a_0 - a_n$$

So if  $a_n \rightarrow L$  as  $n \rightarrow \infty$ , then

$$\sum_{k=0}^{\infty} (a_k - a_{k+1}) = a_0 - L$$

### Example (iv)

$$\sum_{i=2}^{\infty} \left( \frac{1}{i+1} - \frac{1}{i-1} \right)$$

$$S_1 = \frac{1}{3} - 1$$

$$S_2 = \left( \frac{1}{3} - 1 \right) + \left( \frac{1}{4} - \frac{1}{2} \right)$$

$$S_3 = \left(\frac{1}{3} - 1\right) + \left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{1}{5} - \frac{1}{3}\right)$$

$$= -1 + \frac{1}{4} - \frac{1}{2} + \frac{1}{5}$$

$$S_4 = S_3 + \left(\frac{1}{6} - \frac{1}{4}\right)$$

$$= -1 - \frac{1}{2} + \frac{1}{5} + \frac{1}{6}$$

$$S_5 = S_4 + \left(\frac{1}{7} - \frac{1}{5}\right) = -1 - \frac{1}{2} + \frac{1}{6} + \frac{1}{7}$$

$$S_n = -1 - \frac{1}{2} + \frac{1}{n+1} + \frac{1}{n+3}$$

$$S_n \rightarrow -\frac{3}{2} \text{ as } n \rightarrow \infty$$

so

$$\sum_{i=2}^{\infty} \left(\frac{1}{i+1} - \frac{1}{i-1}\right) = -\frac{3}{2}$$

### The Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

The partial sums:

$$S_1 = 1$$

$$S_2 = 1 + \frac{1}{2}$$

$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{2}{2}$$

$$S_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 1 + \frac{3}{2}$$

$$S_{2^n} > 1 + \frac{n}{2} \rightarrow \infty \text{ as } n \rightarrow \infty$$

so

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

Note that the Harmonic series has terms that go to zero, but the sum of those terms go to infinity. This is similar to

$$\int_1^{\infty} \frac{1}{x} dx = \infty$$

from section 7.8.

Theorem (Test for Divergence)

If  $\sum_{n=1}^{\infty} a_n$  converges, then  $\lim_{n \rightarrow \infty} a_n = 0$ .

That is, if  $(a_n \rightarrow 0 \text{ as } n \rightarrow \infty)$  is false, then  $(\sum_{n=1}^{\infty} a_n \text{ converges})$  is also false.

Divergence Test:

$$\lim_{n \rightarrow \infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges}$$

Example (v)  $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$  show the series diverges.

Since  $y = \lim_{x \rightarrow \infty} \left(1 - \frac{1}{x}\right)^x \Leftrightarrow \ln(y) = \lim_{x \rightarrow \infty} x \ln\left(1 - \frac{1}{x}\right)$

let  $u = \frac{1}{x}$   $\Leftrightarrow \ln(y) = \lim_{u \rightarrow 0^+} \frac{\ln(1-u)}{u} \stackrel{\text{LHR}}{\Leftrightarrow} \ln(y) = \lim_{u \rightarrow 0^+} \frac{-\frac{1}{1-u}}{1} = -1$

so  $y = e^{-1}$

$\therefore \left(1 - \frac{1}{n}\right)^n \rightarrow e^{-1} \neq 0 \text{ as } n \rightarrow \infty$

$\therefore$  the series diverges by the Divergence Test.

Example (vi)  $\sum_{n=0}^{\infty} (-1)^n$ . Converges or diverges?

$\lim_{n \rightarrow \infty} (-1)^n = \text{DNE}$ ,  $\therefore$  the series diverges by the Divergence Test.

Note: the partial sums are

$$\text{so } s_1 = 1, s_2 = 0, s_3 = 1, s_4 = 0, \dots$$
$$\lim_{n \rightarrow \infty} s_n = \text{DNE}.$$

Example (vii)  $\sum_{n=0}^{\infty} \frac{2^n}{3^n + n}$ . Apply Divergence test.

$$\lim_{n \rightarrow \infty} \frac{2^n}{3^n + n} = 0, \text{ so the Divergence test}$$

tells us nothing about the convergence/divergence of this series.

Example (viii)  $\sum_{n=0}^{\infty} \frac{4^n}{2^n + 3^n}$

$$\lim_{n \rightarrow \infty} \frac{4^n}{2^n + 3^n} = \infty, \therefore \text{the series diverges by the Divergence Test.}$$

Note the Divergence Test will never conclude that a series converges.