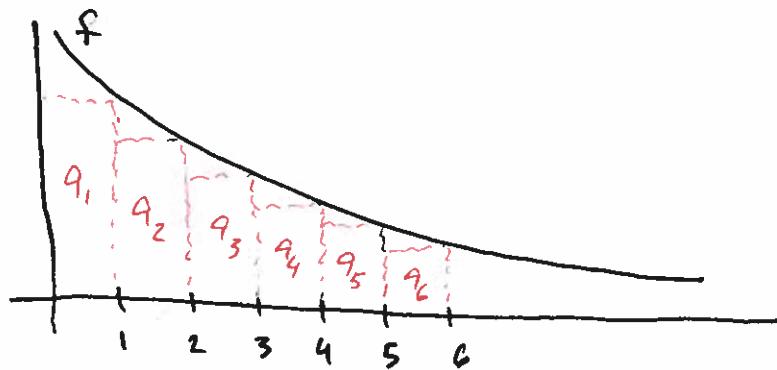


§11.3 Integral Test and Estimates of Sums

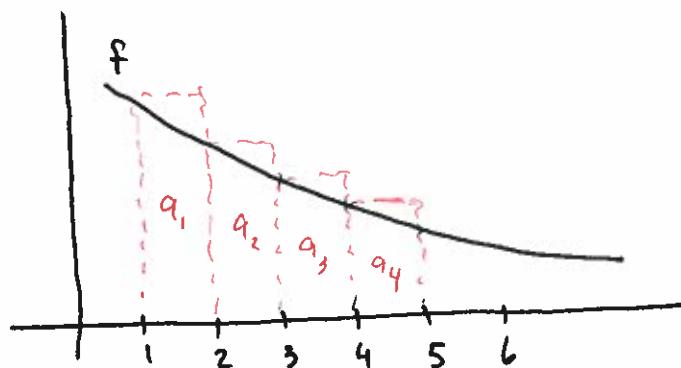
Suppose $f(x)$ is a positive, decreasing function and $a_n = f(n)$.



$$\sum_{k=2}^{\infty} a_k = a_2 + a_3 + a_4 + \dots \leq \int_1^{\infty} f(x) dx$$

$$\sum_{k=1}^{\infty} a_k \leq a_1 + \int_1^{\infty} f(x) dx$$

So if $\int_1^{\infty} f(x) dx$ converges, then $\sum_{k=1}^{\infty} a_k$ also converges.



$$\int_1^\infty f(x)dx \leq \sum_{k=1}^\infty a_k$$

So if $\int_1^\infty f(x)dx$ diverges (to infinity), then $\sum_{k=1}^\infty a_k$ diverges also.

This is the Integral test.

Recall that in section 7.8 we saw

~~$$\int_1^\infty \frac{1}{x^p} dx = \begin{cases} \text{conv.}, & \text{if } p > 1 \\ \text{div.}, & \text{if } p \leq 1 \end{cases}$$~~

P-series

$$\sum_{n=1}^\infty \frac{1}{n^p} = \begin{cases} \text{conv.}, & \text{if } p > 1 \\ \text{div.}, & \text{if } p \leq 1 \end{cases}$$

By the Integral test.

Example (i)

$$\sum_{n=2}^\infty \frac{1}{n \ln(n)}$$

$f(x) = \frac{1}{x \ln(x)}$ is a positive, decreasing function for $x \geq 2$ (graph it).

$$\begin{aligned} & \int_2^\infty \frac{1}{x \ln(x)} dx & u = \ln(x) \\ & \quad du = \frac{1}{x} dx & \\ & = \int_{\ln(2)}^\infty \frac{1}{u} du & \text{which diverges by} \\ & & \text{the P-test for Integrals} \end{aligned}$$

\therefore the series diverges by the Integral test

Example (ii) $\sum_{n=1}^{\infty} n e^{-n^2}$

$f(x) = x e^{-x^2}$ is positive, decreasing (for large x)

$$\begin{aligned} & \int_1^{\infty} x e^{-x^2} dx \quad u = -x^2 \\ &= -\frac{1}{2} \int_{-1}^{-\infty} e^u du \quad du = -2x dx \\ &= \frac{1}{2} \lim_{t \rightarrow -\infty} \int_t^{-1} e^u du \\ &= \frac{1}{2} \lim_{t \rightarrow -\infty} e^u \Big|_t^{-1} \\ &= \frac{1}{2} \lim_{t \rightarrow -\infty} (e^{-t} - e^t) = \frac{1}{2} e^{-1} \text{ (converges)} \end{aligned}$$

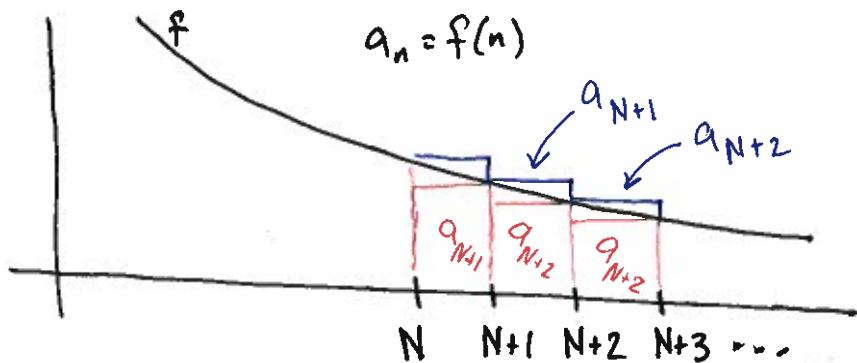
\therefore The series converges by the Integral test.

Example (iii) $\sum_{k=0}^{\infty} \frac{1}{k^2 + 1}$

$$\int_0^{\infty} \frac{1}{x^2 + 1} dx = \tan^{-1}(x) \Big|_0^{\infty} = \frac{\pi}{2} \text{ (converges)}$$

\therefore The series converges by the Integral test.

Remainder Estimate



Let $S = \sum_{n=1}^{\infty} a_n$, $S_N = \sum_{n=1}^N a_n$, $R_N = \sum_{n=N+1}^{\infty} a_n$

then

$$S = S_N + R_N$$

↑
Remainder

N^{th} partial sum (estimate for S)

$$\int_{N+1}^{\infty} f(x) dx \leq R_N \leq \int_N^{\infty} f(x) dx$$

Example (iv)

Estimate $\sum_{n=1}^{\infty} \frac{1}{n^2}$ accurate to 10^{-3} .

We want to find N so that $R_N < 10^{-3}$.

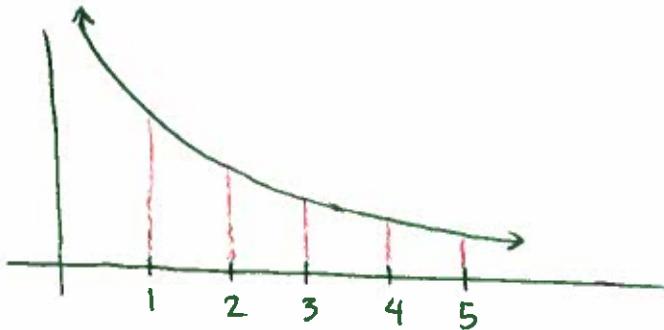
so look at

$$\int_N^{\infty} \frac{1}{x^2} dx = -\frac{1}{x} \Big|_N^{\infty} = \frac{1}{N} \stackrel{\text{set}}{<} 10^{-3}$$

so $N > 1000$. Thus $\sum_{n=1}^{1001} \frac{1}{n^2} \stackrel{\text{CAS}}{\approx} 1.64394$

Example (v)

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Find N so that $\sum_{n=1}^N \frac{1}{n} > 100$



$$\int_1^{N+1} \frac{1}{x} dx \leq \sum_{n=1}^N \frac{1}{n} \leq 1 + \int_1^N \frac{1}{x} dx$$

$$\ln(N+1) \leq \sum_{n=1}^N \frac{1}{n} \leq 1 + \ln(N)$$

So set $100 < \ln(N+1)$.

then $N+1 > e^{100}$ or $N \geq e^{100} \approx 2.688 \times 10^{43}$

Note that it would take a long time to calculate $\sum_{n=1}^{2.7 \times 10^{43}} \frac{1}{n}$

How Long? Suppose we have a machine that could accurately add one trillion terms per second. Then it would take about

2.7×10^{31} seconds ≈ 4.5 minutes \approx

7.5×10^{27} hours $\approx 3.1 \times 10^{26}$ days \approx

8.5×10^{23} years $\approx 6.55 \times 10^{13}$ age of universe.

Don't try this in your calculator!!!