

§11.4 The Comparison Tests

For this section assume $\{a_n\}$ and $\{b_n\}$ are positive sequences.

Comparison test

If $\exists N \in \mathbb{Z}^+ \ni a_n \leq b_n$ for all $n \geq N$ and

- (1) $\sum b_n$ converges, then $\sum a_n$ converges.
- (2) $\sum a_n$ diverges, then $\sum b_n$ diverges.

Example (i)

$$\sum_{n=0}^{\infty} \frac{1}{3^n + 1}$$

$\frac{1}{3^n + 1} < \frac{1}{3^n}$, $\forall n \geq 0$ and $\sum_{n=0}^{\infty} \frac{1}{3^n}$ converges

b/c it is a geometric series with $r = \frac{1}{3}$ ($|r| < 1$). \therefore the series converges by the comparison test.

Example (ii)

$$\sum_{n=2}^{\infty} \frac{n}{n^2 - 1}$$

$\frac{n}{n^2 - 1} > \frac{n}{n^2} = \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (Harmonic series). \therefore the series diverges by the comparison test.

Example (iii) $\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^3}$

$\cos^2(n) \leq 1$, so $\frac{\cos^2(n)}{n^3} \leq \frac{1}{n^3}$ and

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by the P-test ($p=3 > 1$).

\therefore the series converges by the comparison test.

Limit Comparison test

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms and

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L \text{ with } 0 < L < \infty,$$

then either both series converge or both series diverge.

Example (iv) $\sum_{n=1}^{\infty} \frac{3n^2 + 2n + 1}{\sqrt{4n^7 - n + 1}}$

Let $a_n = \frac{3n^2 + 2n + 1}{\sqrt{4n^7 - n + 1}}$

and $b_n = \frac{n^2}{\sqrt{n^7}} = \frac{1}{n^{3/2}}$, then $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{3}{14} = \frac{3}{2}$

and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by the p-test ($p=\frac{3}{2} > 1$).

\therefore the series converges by the Limit comparison test

* ignore the "small" stuff.

Example (v)

$$\sum_{n=0}^{\infty} \frac{6^n}{2^n + 3^n + 4^n}$$

$$\lim_{n \rightarrow \infty} \frac{6^n}{2^n + 3^n + 4^n} = \infty \neq 0. \therefore \text{the series diverges}$$

by the Divergence test.

Remember that we still have the Divergence and Integral tests.

Example (vi)

$$\sum_{n=1}^{\infty} \frac{1}{2^n - n^2 + 1}$$

$2^n \rightarrow \infty$ faster than $n^2 \rightarrow \infty$, so let $b_n = \frac{1}{2^n}$ (ignore the "small" stuff). $a_n = \frac{1}{2^n - n^2 + 1}$,

$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1$ and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges by Geometric Series ($r = \frac{1}{2} < 1$). \therefore The series converges by the limit comp. test.

Example (vii)

$$\sum_{n=2}^{\infty} \frac{1}{n (\ln(n))^2}$$

$$\int_2^{\infty} \frac{1}{x (\ln(x))^2} dx \stackrel{u=\ln(x)}{=} \int_{\ln(2)}^{\infty} \frac{1}{u^2} du = \text{converges by p-test}$$

for integrals. \therefore the series converges by the Integral test.

Remember that we still have earlier tests!

Note that in the previous example

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{n(\ln(n))^2}}{\frac{1}{n^p}} = \begin{cases} \infty, & p > 1 \\ 0, & p \leq 1 \end{cases}$$

so we can not apply the Limit comparison test. Also

$$\frac{1}{n(\ln(n))^2} > \frac{1}{n^p} \text{ for } p > 1 \text{ and } n \text{ large}$$

$$\frac{1}{n(\ln(n))^2} < \frac{1}{n^p} \text{ for } p \leq 1 \text{ and } n \text{ large}$$

so the inequalities are wrong to apply the comparison test.

Example (viii)

$$\sum_{n=1}^{\infty} \frac{2 + \cos(n)}{n}$$

The numerator is bounded and the denominator goes to infinity "slowly", so initial guess is diverges:

$$-1 \leq \cos(n) \leq 1 \Rightarrow 1 \leq 2 + \cos(n) \leq 3 \Rightarrow$$

$\frac{1}{n} \leq \frac{2 + \cos(n)}{n} \leq \frac{3}{n}$, Use the left inequality
because $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (Harmonic series).

∴ the series diverges by the comparison test.