

§11.5 Alternating Series

Let $\{b_n\}$ be a positive, decreasing sequence and $\lim_{n \rightarrow \infty} b_n = 0$, then

$$\sum_{k=0}^{\infty} (-1)^k b_k = b_0 - b_1 + b_2 - b_3 + \dots$$

converges. This is the Alternating Series test.

Note: $0 < s_1 = b_0$

$$0 < s_3 = b_0 - b_1 + b_2 < b_0$$

$$0 < s_5 = b_0 - b_1 + b_2 - b_3 + b_4 < b_0$$

$$0 < s_{2n+1} < b_0$$

Also $s_1 > s_3 > s_5 > \dots$

So by the Monotone Convergence theorem

$$s_{2n+1} \rightarrow s \text{ for some } s \in \mathbb{R}$$

Now look at

$$s_{2n+1} - s_{2n} = b_{2n} .$$

Let $n \rightarrow \infty$ to get

$$s - \lim_{n \rightarrow \infty} s_{2n} = 0$$

or

$$\lim_{n \rightarrow \infty} s_{2n} = s$$

Since $s_{2n} \rightarrow s$ and $s_{2n+1} \rightarrow s$, then $s_n \rightarrow s$ as $n \rightarrow \infty$.

Example (i) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

This is the Alternating Harmonic series.

Note $b_n = \frac{1}{n}$, so $b_n > 0 \ \forall n$, $\{b_n\}$ is decreasing ($f(x) = \frac{1}{x} \Rightarrow f'(x) = -\frac{1}{x^2} < 0 \ \forall x$), and $b_n \rightarrow 0$ as $n \rightarrow \infty$. \therefore the series converges by the Alt. Ser. test.

Example (ii) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot n}{2n-1}$

$b_n = \frac{n}{2n-1}$. $b_n > 0$ for $n = 1, 2, 3, \dots$. Since

$f(x) = \frac{x}{2x-1} \Rightarrow f'(x) = \frac{-1}{(2x-1)^2} < 0 \ \forall x$, then $\{b_n\}$ is decreasing. But

$$\lim_{n \rightarrow \infty} b_n = \frac{1}{2} \neq 0$$

\therefore the series diverges by the Divergence test.

Note: the Alt. Ser. Test will never conclude that a series diverges.

Example (iii) $\sum_{k=0}^{\infty} \frac{(-1)^k \cos(k)}{2k^2+1}$

If $b_k = \frac{\cos(k)}{2k^2+1}$, then $b_k > 0 \ \forall k$ is not true.

So the Alt. Ser. test does not apply. The terms are not all positive (or the same sign), so the comparison tests do not apply, nor does the Integral test. We will look at this one again in a later section.

Alternating Series Estimate

Let $s = \sum_{k=0}^{\infty} (-1)^k b_k$ where $\{b_k\}$ is a positive, decreasing sequence with limit zero. So the series converges by the Alt. Ser. test. We can write $s = s_N + R_N$ where

$$s_N = \sum_{k=0}^{N-1} (-1)^k b_k \quad \text{and} \quad R_N = \sum_{k=N}^{\infty} (-1)^k b_k$$

Without loss of generality, look at N even, then

$$R_N = b_N - \underbrace{b_{N+1} + b_{N+2} - b_{N+3} + b_{N+4}}_{< 0} - \dots$$

$$\text{So } |R_N| < b_N.$$

Thus the error in approximating s by s_N is bounded by b_N (the first term left off in the sum).

Example (iv) Estimate $s = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$ accurate to 10^{-6} .

By the Alt. Ser. Estimate $a = \sum_{k=1}^{N-1} \frac{(-1)^{k+1}}{k^2}$ approximates s with an error bounded by $\frac{1}{N^2}$. So set

$$\frac{1}{N^2} < 10^{-6} \Rightarrow N^2 > 10^6 \Rightarrow N > 10^3 \Rightarrow N=1001 \text{ works}$$

$$\text{So } a = \sum_{k=1}^{1000} \frac{(-1)^{k+1}}{k^2} \approx 0.8224665$$

Example (v) We will see in a later section that $\sin(1) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}$. Use this to estimate $\sin(1)$ accurate to 10^{-6} .

Use the Alternating Series Estimate

$$\begin{aligned}\sin(1) &= 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \frac{1}{9!} - \frac{1}{11!} + \frac{1}{13!} - \dots \\ &\approx \underbrace{1 - \frac{1}{6} + \frac{1}{120}}_{\text{approximation}} - 1.98 \times 10^{-4} + 2.76 \times 10^{-6} - \underbrace{2.5 \times 10^{-8}}_{< 10^{-6}} + \dots\end{aligned}$$

so

$$\sin(1) \approx \sum_{k=0}^4 \frac{(-1)^k}{(2k+1)!} \approx 0.841471$$

with error bounded by approximately 2.5×10^{-8} .