

§11.6 Absolute convergence, Ratio and Root Tests

Definition $\sum_{n=1}^{\infty} |a_n|$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

$\sum_{n=1}^{\infty} a_n$ is conditionally convergent if it converges, but is not absolutely convergent.

Theorem If $\sum_{n=1}^{\infty} |a_n|$ converges absolutely, then it converges.

Example (i) $\sum_{k=0}^{\infty} \frac{(-1)^k \cos(k)}{2k^2 + 1}$

$$\left| \frac{(-1)^k \cos(k)}{2k^2 + 1} \right| \leq \frac{1}{2k^2 + 1} \stackrel{\text{let}}{=} a_k. \text{ Let } b_k = \frac{1}{k^2}.$$

$\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = \frac{1}{2}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by the

p-test ($p=2 > 1$), so $\sum_{k=0}^{\infty} \frac{1}{2k^2 + 1}$ converges by

the limit comparison test. \therefore the series converges absolutely by the comparison test.

(Remember that the first few terms don't matter.

So k can start with different values when applying ~~the~~ comparison tests.)

Example(ii) $s = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges conditionally.

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges (Harmonic series)}$$

So s does not converge absolutely, but the Alternating Series test implies s converges. So the convergence is conditional.

Ratio Test

Let $s = \sum_{n=1}^{\infty} a_n$. If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$ and

- (1) $L < 1$, then s converges absolutely.
- (2) $L > 1$, then s diverges.
- (3) $L = 1$, then the test fails.

Root test

Let $s = \sum_{n=1}^{\infty} a_n$. If $\lim_{n \rightarrow \infty} |a_n|^{1/n} = L$ and

- (1) $L < 1$, then s converges absolutely
- (2) $L > 1$, then s diverges
- (3) $L = 1$, then the test fails.

Note that these two tests are equivalent. If one of the tests fails, then so will the other.

The Ratio test works well on series that have factorials in the terms. The Root test works well on series that have exponentials, but not factorials in the terms.

Example (iii) $\sum_{n=1}^{\infty} \frac{1}{n}$

Ratio test: $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{1}{n+1}}{\frac{1}{n}} \right| = 1$

Root test: $\lim_{n \rightarrow \infty} |a_n|^{1/n} = \lim_{n \rightarrow \infty} \left(\frac{1}{n} \right)^{1/n} = 1$

$$(y = \lim_{n \rightarrow \infty} \left(\frac{1}{n} \right)^{1/n} \Rightarrow \ln(y) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln\left(\frac{1}{n}\right))$$

$$\Rightarrow \ln(y) = \lim_{n \rightarrow \infty} -\frac{\ln(n)}{n} \xrightarrow{\text{LHR}} \ln(y) = \lim_{n \rightarrow \infty} -\frac{\frac{1}{n}}{1} = 0$$
$$\Rightarrow y = e^0 = 1$$

Example (iv) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Ratio test: $\lim_{n \rightarrow \infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = 1$

Root test: $\lim_{n \rightarrow \infty} \left(\frac{1}{n^2} \right)^{1/n} = 1$

These two examples show that it is possible for a divergent series to have $L=1$ and also a convergent series to have $L=1$, that is why $L=1 \Rightarrow$ test fails.

Also, by properties of limits, if $P_k(n)$ is any polynomial in the variable n , then

$$(|P_k(n)|)^{1/n} \rightarrow 1 \text{ as } n \rightarrow \infty$$

Example (v) $\sum_{n=0}^{\infty} \frac{(-2)^n}{n!}$

Ratio test : $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-2)^{n+1}}{(n+1)!} \div \frac{(-2)^n}{n!} \right|$
 $= \lim_{n \rightarrow \infty} \frac{2}{(n+1)!} \cdot \frac{n!}{2^n} = \lim_{n \rightarrow \infty} \frac{2}{n+1} = 0 < 1$

\therefore the series converges absolutely by the Ratio test.

Note: $(n+1)! = (n+1)(n!),$ so $\frac{n!}{(n+1)!} = \frac{1}{n+1}.$

Example (vi) $\sum_{n=0}^{\infty} \frac{3^n (n!)^2}{(2n)!}$

Ratio test: $\lim_{n \rightarrow \infty} \frac{3^{n+1} ((n+1)!)^2}{(2(n+1))!} \cdot \frac{(2n)!}{3^n (n!)^2}$
 $= \lim_{n \rightarrow \infty} \frac{3 (n+1)^2}{(2n+2)(2n+1)} = \frac{3}{4} < 1$

\therefore the series conv. abs. by the Ratio test.

Note: $(2n+2)! = (2n+2)(2n+1)((2n)!).$

What if we had $\sum_{n=0}^{\infty} \frac{5^n (n!)^2}{(2n)!}?$ Then the limit would be $\frac{5}{4} > 1$ and the series diverges.

Example (vii) $\sum_{n=1}^{\infty} \frac{n^{1000}}{2^n}$ converges by the Root test.

$$\lim_{n \rightarrow \infty} \left(\frac{n^{1000}}{2^n} \right)^{1/n} = \lim_{n \rightarrow \infty} \frac{(n^{1/n})^{1000}}{2} = \frac{1}{2} < 1.$$

Example (viii) $\sum_{n=0}^{\infty} \frac{n^n}{n^{10} + 2^n}$ diverges by the Root test

$$\lim_{n \rightarrow \infty} \left(\frac{n^n}{n^{10} + 2^n} \right)^{1/n} = \lim_{n \rightarrow \infty} \frac{n}{(n^{10} + 2^n)^{1/n}} = \frac{\infty}{2} > 1$$

Note that we could also say $\frac{n^n}{n^{10} + 2^n} \rightarrow \infty$ as $n \rightarrow \infty$, so the series diverges by the Divergence Test.

Example (ix) $\sum_{n=1}^{\infty} \frac{(-4)^{n-1} n!}{n^n}$

Because of the factorial, use Ratio test.

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left| \frac{(-4)^n (n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{(-4)^{n-1} n!} \right| \\ &= \lim_{n \rightarrow \infty} \frac{4(n+1) n^n}{(n+1)^{n+1}} = \lim_{n \rightarrow \infty} 4 \left(\frac{n}{n+1} \right)^n = \frac{4}{e} > 1 \end{aligned}$$

∴ the series diverges by the Ratio test.

Note $\left(\frac{n+1}{n} \right)^n = \left(1 + \frac{1}{n} \right)^n \rightarrow e$ as $n \rightarrow \infty$,

so $\left(\frac{n}{n+1} \right)^n \rightarrow \frac{1}{e}$ as $n \rightarrow \infty$.