

§ 11.11 Applications of Taylor Polynomials

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3 + \dots$$

and

$$|R_n(x)| = |f(x) - T_n(x)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1} \right|$$

where c is some number between x and a .

Taylor's Inequality

If $|f^{(n+1)}(x)| \leq M_n$ for $|x-a| \leq d$ where M does not depend on x , but could depend on n , then

$$|R_n(x)| \leq \frac{M_n}{(n+1)!} d^{n+1}$$

Linear Approximations (calc I)

$$f(x) \approx T_1(x) = f(a) + f'(a)(x-a)$$

Quadratic Approximation

$$f(x) \approx T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2} (x-a)^2$$

Example (i) Let $f(x) = x^{1/4}$, $a=16$. Find the linear and quadratic approximations. How accurate is the quadratic approximation on the interval $[15, 17]$?

$$f(x) = x^{1/4}$$

$$f(16) = 2$$

$$f'(x) = \frac{1}{4} x^{-3/4}$$

$$f'(16) = \frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32}$$

$$f''(x) = -\frac{3}{16} x^{-7/4}$$

$$f''(16) = -\frac{3}{16} \cdot \frac{1}{128} = -\frac{3}{2048}$$

$$f'''(x) = \frac{21}{64} x^{-11/4}$$

Linear Approximation:

$$T_1(x) = 2 + \frac{1}{32}(x-16)$$

Quadratic Approximation:

$$T_2(x) = 2 + \frac{1}{32}(x-16) - \frac{3}{4096}(x-16)^2$$

$$|R_2(x)| \leq \frac{21}{64} (15)^{-11/4} (1)^3 \cdot \frac{1}{3!} \approx 3.2 \times 10^{-5}$$

on $[15, 17]$. (Look at Mathematica graphs)

Example (ii) Use $T_5(x)$ to approximate $\sin(x)$ on $-0.3 \leq x \leq 0.3$. Find $T_5(12^\circ)$, the actual and theoretical error in this approximation.

$$T_5(x) = T_6(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$T_5\left(\frac{12\pi}{180}\right) = 0.207911694322979$$

$$\begin{aligned}\text{Actual error} &= \left| T_5\left(\frac{12\pi}{180}\right) - \sin\left(\frac{12\pi}{180}\right) \right| \\ &= 3.505219 \times 10^{-9}\end{aligned}$$

$$\begin{aligned}\text{Theoretical error} &= |R_6\left(\frac{12\pi}{180}\right)| \leq \frac{1}{7!} \left(\frac{12\pi}{180}\right)^7 \\ &= 3.50736 \times 10^{-9}\end{aligned}$$

Note that the actual error is less than the bound given by Taylor's Inequality.

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Example (iii) Find $T_6(x)$ and a bound on the error with $f(x) = \sqrt[3]{8+x^2}$, $a=0$, $x \in [0, 2]$. Then use Binomial Series and the Alternating Series Estimate to find the same thing.

Using Mathematica:

$$T_6(x) = 2 + \frac{x^2}{12} - \frac{x^4}{288} + \frac{5x^6}{20736}$$

Note: $T_7(x) = T_6(x)$, so

$$|R_7(x)| \leq \frac{3}{8!} 2^8 \approx 1.90476 \times 10^{-3}$$

Using Binomial Series :

$$(1+x)^{1/3} = 1 + \frac{1}{3}x + \left(\frac{1}{3}\right)\left(-\frac{2}{3}\right) \frac{x^2}{2!} + \left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right) \frac{x^3}{3!} + \dots$$
$$= 1 + \frac{1}{3}x - \frac{2}{3^2 2!} x^2 + \frac{2 \cdot 5}{3^3 3!} x^3 - \frac{2 \cdot 5 \cdot 8}{3^4 4!} x^4 + \dots$$

Then

$$(8+x^2)^{1/3} = 2 \left(1 + \frac{x^2}{8}\right)^{1/3}$$
$$= 2 \left(1 + \frac{1}{3} \left(\frac{x^2}{8}\right)\right) - \frac{2}{3^2 \cdot 2!} \left(\frac{x^2}{8}\right)^2 + \frac{2 \cdot 5}{3^3 \cdot 3!} \left(\frac{x^2}{8}\right)^3 - \frac{2 \cdot 5 \cdot 8}{3^4 \cdot 4!} \left(\frac{x^2}{8}\right)^4 + \dots$$

so $T_6(x) = 2 + \frac{x^2}{12} - \frac{x^4}{288} + \frac{5x^6}{20736}$

Alternating Series Estimate gives

$$|\text{error}| \leq \left| \frac{2 \cdot 2 \cdot 5 \cdot 8}{3^4 \cdot 4!} \frac{x^8}{8^4} \right| \leq 0.00514403$$
$$5.14403 \times 10^{-3}$$

Look at graphs.