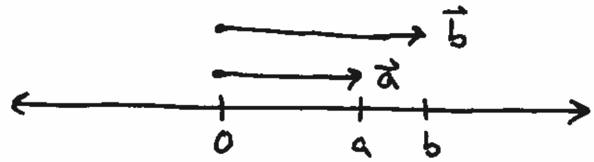
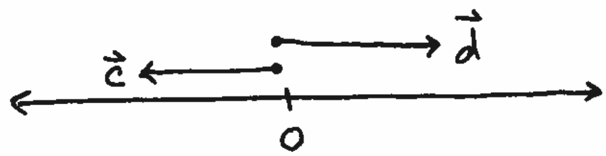
§12.3 The Dot Product

Multiplication of real numbers as vectors

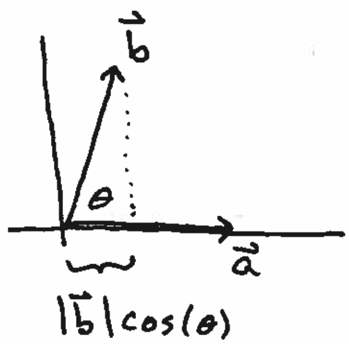


$\vec{a} \circ \vec{b} = |\vec{a}||\vec{b}|$ angle "between" $\vec{a}$ & $\vec{b}$ is 0



$\vec{c} \circ \vec{d} = -|\vec{c}||\vec{d}|$ angle "between" $\vec{c}$ & $\vec{d}$ is π

Now vectors in $\mathbb{R}^2$



Def $\vec{a} \circ \vec{b} = |\vec{a}||\vec{b}|\cos(\theta)$, where θ is the angle between $\vec{a}$ & $\vec{b}$.

Def $\langle a_1, a_2 \rangle \circ \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2$

These two definitions are consistent. That is, it can be shown that they produce the same value. We can extend these definitions to $\mathbb{R}^n$. The second definition becomes

$$\vec{a} \circ \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 + \dots + a_n b_n$$

This is the dot product.

<u>Def</u> The angle between $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^n$ is given by

$$\theta = \cos^{-1}\left(\frac{\vec{a} \circ \vec{b}}{|\vec{a}|\,|\vec{b}|}\right)$$

This comes from combining the two definitions.

<u>Example</u>

Find the angle between $\langle 1,2,3 \rangle$ and $\langle 5,-2,7 \rangle$

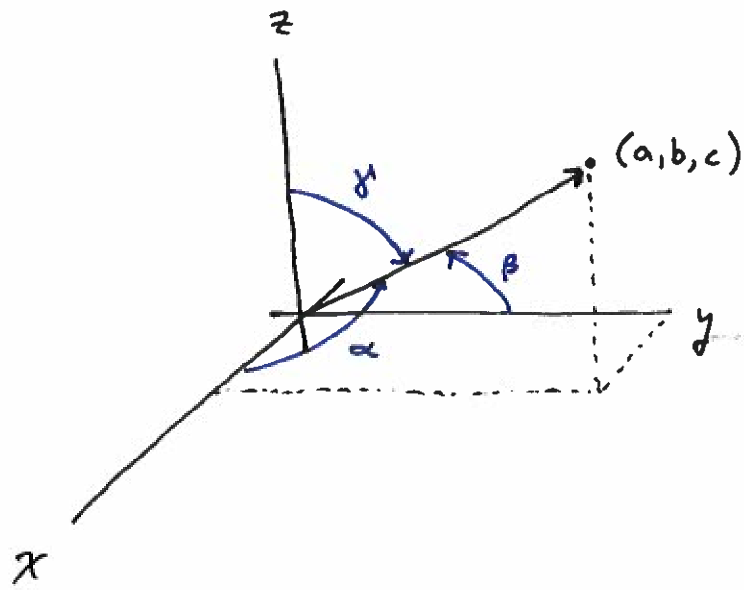
$$\theta = \cos^{-1}\left(\frac{\langle 1,2,3\rangle \circ \langle 5,-2,7\rangle}{|\langle 1,2,3\rangle|\,|\langle 5,-2,7\rangle|}\right)$$

$$= \cos^{-1}\left(\frac{(1)(5) + (2)(-2) + (3)(7)}{\sqrt{1+4+9}\,\sqrt{25+4+49}}\right)$$

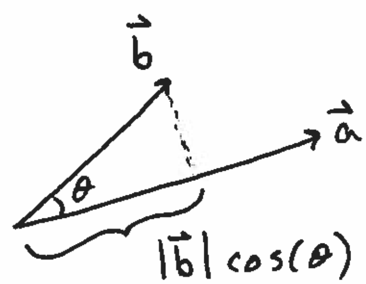
$$= \cos^{-1}\left(\frac{\overset{22}{\cancel{20}}}{\sqrt{14}\,\sqrt{78}}\right)$$

$$= \cancel{.9207} \text{ or } \cancel{52.8}^\circ \quad .8423 \text{ or } 48.3^\circ$$

Direction Angles



Comp and Proj



$$|\vec{b}|\cos(\theta) = \frac{|\vec{a}||\vec{b}|\cos(\theta)}{|\vec{a}|}$$

<u>Def</u> the component of $\vec{b}$ in the direction of $\vec{a}$ is given by

$$\text{Comp}_{\vec{a}}(\vec{b}) = \frac{\vec{a}\circ\vec{b}}{|\vec{a}|}$$

Def the projection of $\vec{b}$ onto $\vec{a}$ is the vector given by

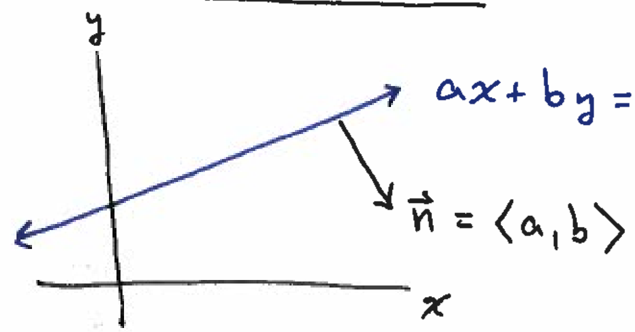
$$\operatorname{proj}_{\vec{a}}(\vec{b}) = \frac{\vec{a}\circ\vec{b}}{|\vec{a}|^2}\,\vec{a}$$

$$= \operatorname{comp}_{\vec{a}}(\vec{b})\cdot\frac{\vec{a}}{|\vec{a}|}$$

These are sometimes called the scalar and vector projections of $\vec{b}$ onto $\vec{a}$ and can be found using the components of $\vec{a}$ and $\vec{b}$ without needing to know the angle between the two vectors.

Note that if $\vec{a}\perp\vec{b}$ (perpedicular, orthogonal, or normal $= \perp$), then $\vec{a}\circ\vec{b} = 0$ and also $\vec{a}\circ\vec{b} = 0 \Rightarrow \vec{a}\perp\vec{b}$ because $\cos(90^\circ) = 0$.
So $\vec{a}\perp\vec{b} \Rightarrow \operatorname{comp}_{\vec{a}}(\vec{b}) = 0$ and $\operatorname{proj}_{\vec{a}}(\vec{b}) = \vec{0}$.

Equations of lines



$ax + by = c \Rightarrow \text{slope} = -\frac{a}{b}$

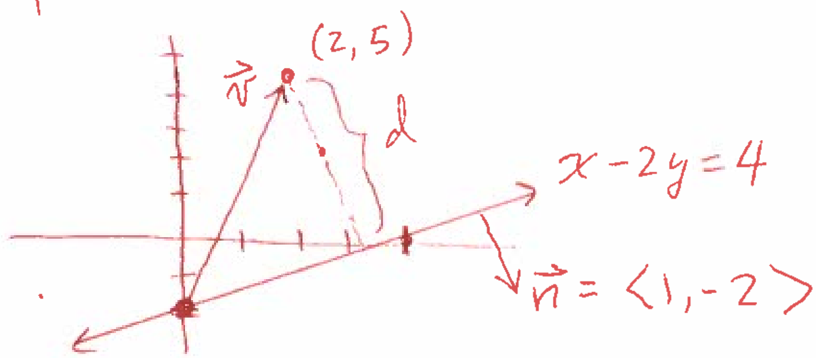
$\Rightarrow \perp \text{ slope} = \frac{b}{a}$

So the line with equation $ax+by=c$ has a normal vector $\vec{n} = \langle a, b\rangle$.

Example

Find the orthogonal distance between the point $(2,5)$ and the line $x-2y=4$.

Draw a picture:



We get $\vec{n}$ from the x and y coeficients of the equation of the line in standard form. Next pick a point on the line, say $(0,-2)$ and form the vector $\vec{v}=\langle 2,7\rangle$ as shown. Then the orthogonal distance is

$$d = \left|\operatorname{comp}_{\vec{n}}(\vec{v})\right|$$

$$= \left|\frac{\langle 1,-2\rangle \cdot \langle 2,7\rangle}{|\langle 1,-2\rangle|}\right|$$

$$= \left|\frac{2-14}{\sqrt{1+4}}\right|$$

$$= \frac{12}{\sqrt{5}}$$

Note that we could have used $\vec{n}=\langle -1,2\rangle$ and then the outer absolute values would not be needed.