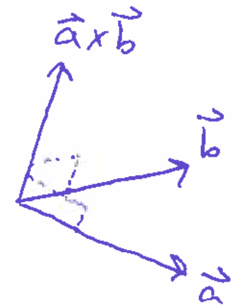
§12.4 Vector Cross-Product

Must be in $\mathbb{R}^3$ (3 dimensional space)

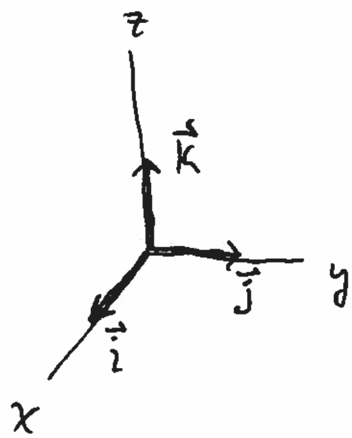
<u>Def</u> $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin(\theta)$

where θ is the angle between $\vec{a}$ & $\vec{b}$.

$\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ & $\vec{b}$ using the right-hand-rule



So



$$\vec{i} \times \vec{k} = -\vec{j} = -(\vec{k} \times \vec{i})$$

$$\vec{j} \times \vec{k} = \vec{i} = -(\vec{k} \times \vec{j})$$

$$\vec{i} \times \vec{j} = \vec{k} = -(\vec{j} \times \vec{i})$$

$$\vec{a} \times \vec{a} = \vec{0}, \underbrace{\forall \vec{a} \in \mathbb{R}^3}_{\text{for all } \vec{a} \text{ in } \mathbb{R}^3}$$

Assuming distribution works (and it does), then

$$(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \times (b_1\vec{i} + b_2\vec{j} + b_3\vec{k}) =$$

$$(a_1\vec{i})\times(b_1\vec{i}) + (a_1\vec{i})\times(b_2\vec{j}) + (a_1\vec{i})\times(b_3\vec{k})$$
$$+ (a_2\vec{j})\times(b_1\vec{i}) + (a_2\vec{j})\times(b_2\vec{j}) + (a_2\vec{j})\times(b_3\vec{k})$$
$$+ (a_3\vec{k})\times(b_1\vec{i}) + (a_3\vec{k})\times(b_2\vec{j}) + (a_3\vec{k})\times(b_3\vec{k}) =$$

$$\vec{0} + a_1b_2\vec{k} - a_1b_3\vec{j} - a_2b_1\vec{k} + \vec{0} + a_2b_3\vec{i}$$
$$+ a_3b_1\vec{j} - a_3b_2\vec{i} + \vec{0} =$$

$$\langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \rangle$$

Matrices and Determinant

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For larger matrices, we can "Expand by Cofactors"

For 3x3 matrices:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

(look determinant up on the web for more details)

Then $\langle a_1, a_2, a_3 \rangle \times \langle b_1, b_2, b_3 \rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Example: Find $\langle 1,2,3\rangle \times \langle 2,-5,7\rangle$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 2 & -5 & 7 \end{vmatrix} = \vec{i}\begin{vmatrix} 2 & \overset{3}{\cancel{1}} \\ -5 & \underset{7}{\cancel{8}} \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 2 \\ 2 & -5 \end{vmatrix}$$

$$= \left\langle (2)(\overset{7}{\cancel{1}}) - (\overset{3}{\cancel{1}})(-5),\ -\big((1)(7)-(2)(3)\big),\ (1)(-5)-(2)(2) \right\rangle$$

$$= \left\langle \overset{14-(-15)}{\cancel{6+35}},\ -(7-6),\ -5-4 \right\rangle$$

$$= \left\langle \overset{29}{\cancel{41}},\ -1,\ -9 \right\rangle$$

Example: Find $\langle 1,2\rangle \times \langle 3,-1\rangle$

Since cross product must occur in $\mathbb{R}^3$, we really look at $\langle 1,2,0\rangle \times \langle 3,-1,0\rangle$

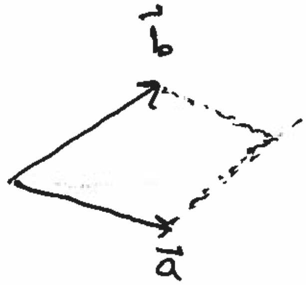
$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 3 & -1 & 0 \end{vmatrix} = \vec{k}\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = 0 + 0$$

$$= \langle 0, 0, -1-6\rangle$$

$$= \langle 0, 0, -7\rangle$$

Note we expand by the last column b/c of the zero's.

Area of parallelograms



The area of a parallelogram induced by two vectors $\vec{a}$ & $\vec{b}$ is $|\vec{a} \times \vec{b}|$

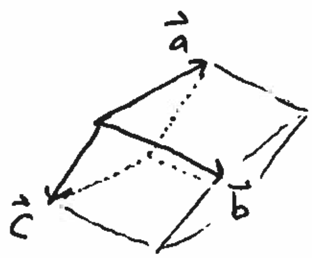
Example: Find the area of the parallelogram induced by $\langle 1,2 \rangle$ and $\langle 3,-1 \rangle$

$$A = |\vec{a} \times \vec{b}| = |\langle 1,2,0 \rangle \times \langle 3,-1,0 \rangle|$$

see previous example

$$A = |\langle 0,0,-7 \rangle| = 7$$ square units

Volume of a Parallelepiped

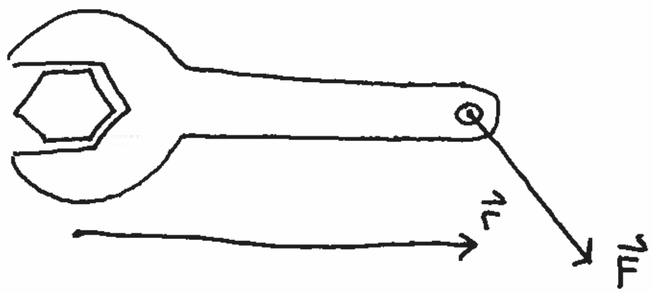


Three vectors induce a parallelepiped. The Volume is given by

$$|\vec{a} \times \vec{b} \circ \vec{c}|$$

Note that the <u>ONLY</u> way to calculate $\vec{a} \times \vec{b} \circ c$ is $(\vec{a} \times \vec{b}) \circ \vec{c}$. That is, do the cross product first.

Torque



$\vec{r}$ is the moment arm

$\vec{F}$ is the force

$\vec{\tau} = \vec{r} \times \vec{F}$ is the Torque

Other thoughts

$\vec{a}$ and $\vec{b}$ are parallel $\Leftrightarrow$ $\vec{a} \times \vec{b} = \vec{0}$, provided $\vec{a} \neq \vec{0}$ and $\vec{b} \neq \vec{0}$

Properties are on page 836.

Note: cross product is not communitive

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

so order matters.