§13.2 Derivatives and Integrals of Vector Functions

Definitions: Let $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ be a vector function, then

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h \to 0} \frac{1}{h}\left(\vec{r}(t+h) - \vec{r}(t)\right)$$

provided the limit exists.

$$\int \vec{r}(t)\,dt = \vec{R}(t) + \vec{C}$$

where $\vec{R}\,'(t) = \vec{r}(t)$ and $\vec{C}$ is an arbitrary constant vector

$$\int_a^b \vec{r}(t)\,dt = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \vec{r}(t_k^*)\,\Delta t_k = \vec{R}(b) - \vec{R}(a)$$

where P is a partition of $[a,b]$, $\|P\|$ is the "size" of the partition, t_k^* are representative t values and $\vec{R}\,'(t) = \vec{r}(t)$. (provided the limit exists).

Since sum's and limits of vector functions are performed component-wise, we have

$$\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

and

$$\int \vec{r}(t)\,dt = \left\langle \int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt \right\rangle$$

See page 874 for Differentiation Rules.

Properties for Integrals are also the same as in single variable calculus.

Examples

(i) Find $\vec{r}\,'(t)$ if $\vec{r}(t) = \langle t^3, \tan^{-1}(t), \cosh(t) \rangle$.

$$\vec{r}\,'(t) = \left\langle 3t^2, \frac{1}{1+t^2}, \sinh(t) \right\rangle$$

(ii) Find $\vec{I} = \int \left\langle t^2 e^t, \frac{1}{\sqrt{1-t^2}}, \frac{t^2}{(t-1)^2(t^2+1)} \right\rangle dt$.

The first component is integration by parts:

Tabular form:

$$\begin{array}{ccc} t^2 & + & e^t \\ 2t & - & e^t \\ 2 & + & e^t \\ 0 & & e^t \end{array}$$

$$I_1 = \int t^2 e^t \, dt = t^2 e^t - 2t\,e^t + 2e^t + C_1$$

The second component is a form (see reference page 6 for the 20 "basic" forms)

$$I_2 = \int \frac{1}{\sqrt{1-t^2}} \, dt = \sin^{-1}(t) + C_2$$

The third component is a partial fraction decomposition problem.

$$\frac{t^2}{(t-1)^2(t^2+1)} = \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{Ct+D}{t^2+1}$$

$$\Longleftrightarrow \quad t^2 = A(t-1)(t^2+1) + B(t^2+1) + (Ct+D)\underbrace{(t-1)^2}_{(t^2-2t+1)}$$

Two ways of solving for A, B, C, D:

<u>Substitution</u>

Let $t=1 \Rightarrow \quad 1 = 2B$

Let $t=0 \Rightarrow \quad 0 = -A + B + D$

Let $t=-1 \Rightarrow \quad 1 = -4A + 2B - 4C + 4D$

Let $t=2 \Rightarrow \quad 4 = 5A + 5B + 2C + D$

<u>Equating Coefficients</u>

$t^3 \Rightarrow \quad 0 = A + C$

$t^2 \Rightarrow \quad 1 = -A + B - 2C + D$

$t^1 \Rightarrow \quad 0 = A + C - 2D$

$t^0 \Rightarrow \quad 0 = -A + B + D$

Either way produces a system of equations that gives:

$$A = \frac{1}{2}, \quad B = \frac{1}{2}, \quad C = -\frac{1}{2}, \quad D = 0$$

So

$$I_3 = \frac{1}{2}\int \frac{1}{t-1}\,dt + \frac{1}{2}\int \frac{1}{(t-1)^2}\,dt - \frac{1}{2}\int \frac{t}{t^2+1}\,dt$$

$$= \frac{1}{2}\ln|t-1| - \frac{1}{2}\cdot\frac{1}{t-1} - \frac{1}{4}\ln(t^2+1) + C_3$$

Finally

$$\vec{I} = \langle I_1, I_2, I_3 \rangle$$

note that the arbitrary constant vector is incorporated into each component.

Basically, there is not much new material in this section.