# §13.4 Motion in Space: Velocity and Acceleration

If $\vec{r}(t)$ is position, then velocity is given by $\vec{v}(t) = \vec{r}\,'(t)$ and acceleration is given by $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$.

<u>Example</u> Suppose that $\vec{a}(t) = -\frac{1}{10}\vec{v}(t) - \vec{g}$, with $\vec{v}(0) = \langle 20, 30\rangle$ m/s and $\vec{r}(0) = \langle 0, 0\rangle$. Find $\vec{v}(t)$ and $\vec{r}(t)$.

So $$\vec{v}\,'(t) + \frac{1}{10}\vec{v}(t) = -\langle 0, 9.8\rangle.$$

Multiply by $e^{\frac{1}{10}t}$ to get

$$e^{\frac{1}{10}t}\vec{v}\,'(t) + \frac{1}{10}e^{\frac{1}{10}t}\vec{v}(t) = -e^{\frac{1}{10}t}\langle 0, 9.8\rangle$$

$\Longleftrightarrow$

$$\frac{d}{dt}\left[e^{\frac{1}{10}t}\vec{v}(t)\right] = -e^{\frac{1}{10}t}\langle 0, 9.8\rangle.$$

Integrate both sides:

$$e^{\frac{1}{10}t}\vec{v} = -10\,e^{\frac{1}{10}t}\langle 0, 9.8\rangle + \vec{C}_1.$$

So $$\vec{v}(t) = e^{-\frac{1}{10}t}\vec{C}_1 - \langle 0, 98\rangle.$$

The initial condition $\vec{v}(0) = \langle 20, 30\rangle \Longrightarrow$

$$\langle 20, 30\rangle = \vec{C}_1 - \langle 0, 98\rangle$$

$$\vec{C}_1 = \langle 20, 128\rangle$$

and

$$\boxed{\vec{v}(t) = e^{-\frac{1}{10}t}\langle 20, 128\rangle - \langle 0, 98\rangle}$$

Now $$\vec{r}\,'(t) = e^{-\frac{1}{10}t}\langle 20,128\rangle - \langle 0,98\rangle .$$

Integrate to get

$$\vec{r}(t) = -10e^{-\frac{1}{10}t}\langle 20,128\rangle - t\langle 0,98\rangle + \vec{C}_2 .$$

The initial conditions give

$$\langle 0,0\rangle = \vec{C}_2 - \langle 200,1280\rangle$$

$$\Rightarrow \qquad \vec{C}_2 = \langle 200,1280\rangle$$

and

$$\boxed{\vec{r}(t) = \langle 200,1280\rangle - e^{-\frac{1}{10}t}\langle 200,1280\rangle - t\langle 0,98\rangle .}$$

<u>Tangent and Normal Components of Acceleration</u>

Speed is given by $v(t) = |\vec{v}(t)| = |\vec{r}\,'(t)|$.

So $$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\vec{v}(t)}{v(t)}$$

$$\Rightarrow \qquad \vec{v}(t) = v(t)\vec{T}(t) .$$

Then

$$\vec{a}(t) = v'(t)\vec{T}(t) + v(t)\vec{T}\,'(t) .$$

Recall that $\kappa(t) = \dfrac{|\vec{T}\,'(t)|}{v(t)}$ and $\vec{N} = \dfrac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}$,

so $$\boxed{\vec{a}(t) = v'(t)\vec{T}(t) + \kappa(t)v(t)^2\vec{N}(t)}$$

we write $a_T = v'(t)$ and $a_N = \kappa v^2$

In general, $v'$ and $\kappa v^2$ are difficult to directly calculate. We now will get these quantities in terms of $\vec{r}$:

First note that

$$\vec{v} \cdot \vec{a} = v\vec{T} \cdot (v'\vec{T} + \kappa v^2 \vec{N})$$

$$= v v'$$

so

$$\boxed{a_T = \frac{\vec{v} \cdot \vec{a}}{v} = \frac{\vec{r}' \cdot \vec{r}''}{|\vec{r}'|}}$$

Next recall that $\kappa = \dfrac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3}$

so

$$\boxed{a_N = \kappa v^2 = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|}}$$

<u>Example</u> Let $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$.
Find $a_T$ and $a_N$.

$\vec{r}'(t) = \langle -\sin(t), \cos(t), 1 \rangle$ , $|\vec{r}'(t)| = \sqrt{2}$

$\vec{r}''(t) = \langle -\cos(t), -\sin(t), 0 \rangle$

$\vec{r}' \times \vec{r}'' = \langle \sin(t), -\cos(t), 1 \rangle$ , $|\vec{r}' \times \vec{r}''| = \sqrt{2}$

$\vec{r}' \cdot \vec{r}'' = \sin(t)\cos(t) - \sin(t)\cos(t) + 0 = 0$

so $a_T = 0$ and $a_N = 1$

Thus the acceleration is only in the normal direction.