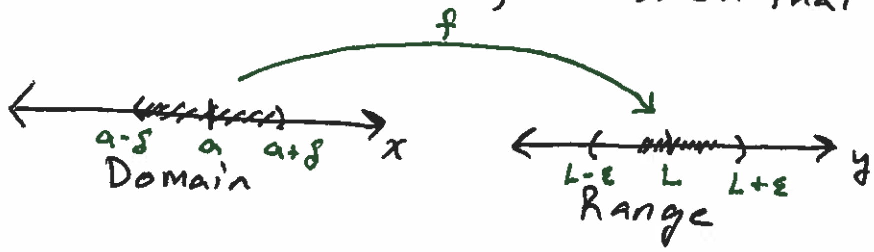
§14.2 Limits and Continuity

Recall: $\lim_{x \to a} f(x) = L \iff$

$\forall \varepsilon > 0, \exists \delta > 0 \ni 0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

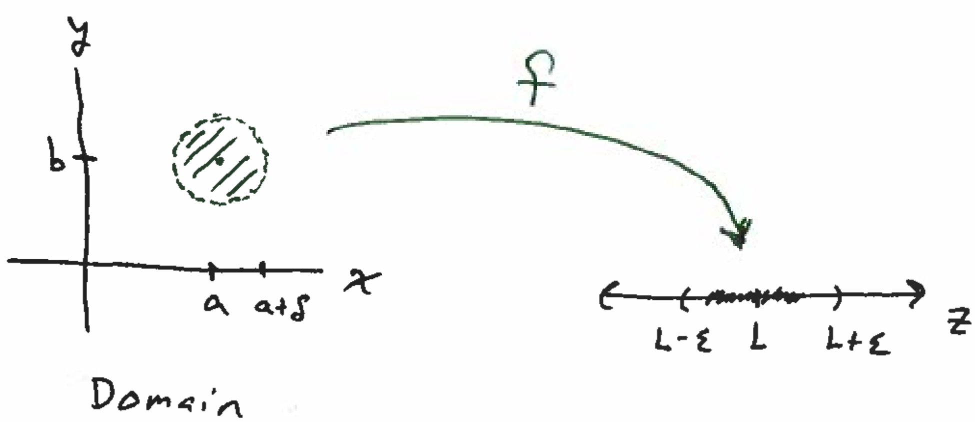
($\forall$: for all, $\exists$: there exists, $\ni$: such that)



Definition Let f be a function of two variables whose domain includes points in an open circle centered at (a,b) with positive radius. Then we say

$$\lim_{(x,y) \to (a,b)} f(x,y) = L \iff$$

$$\forall \varepsilon > 0, \exists \delta > 0 \ni 0 < |\langle x,y \rangle - \langle a,b \rangle| < \delta$$
$$\Rightarrow |f(x,y) - L| < \varepsilon.$$



This definition can be extended to functions of more than two variables.

Alternate Definition $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = L \iff$

$$\lim_{t\to\alpha} f(x(t),y(t)) = L \quad \forall \text{ paths } \vec{r}(t) = \langle x(t), y(t)\rangle$$

where $\displaystyle\lim_{t\to\alpha} \vec{r}(t) = \langle a,b\rangle$.

So, if $f(x,y)\to L_1$ along a path C_1 and $f(x,y)\to L_2$ along another path C_2 where $L_1 \neq L_2$, then $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y)$ Does Not Exist.

Definition f is continuous at (a,b) if

(i) $f(a,b)$ exists

(ii) $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y)$ exists

(iii) $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Polynomials are continuous everywhere. Rational functions and radical functions are continuous on their domains.

Examples

(i) $$\lim_{(x,y)\to(3,-2)} \frac{x+y}{x^2-y^2} = \frac{3+(-2)}{3^2-(-2)^2} = \frac{1}{5}$$

(Note that $(3,-2)$ is in the domain of $z = \frac{x+y}{x^2-y^2}$.)

(ii) $$\lim_{(x,y)\to(0,0)} \frac{1-\cos(\sqrt{x^2+y^2})}{x^2+y^2} = \frac{0}{0}$$

Note that $(0,0)$ is not in the domain, but when we "plug in" we get an indeterminant form. There is no L'HR for functions of two or more Variables! In this case we can change variables: Let $x = r\cos(\theta)$, $y = r\sin(\theta)$, then $r = \sqrt{x^2+y^2}$ and the limit becomes

$$\lim_{r\to 0^+} \frac{1-\cos(r)}{r^2}$$ (Now use L'HR)

$$\overset{LHR}{=} \lim_{r\to 0^+} \frac{\sin(r)}{2r}$$

$$\overset{LHR}{=} \lim_{r\to 0^+} \frac{\cos(r)}{2} = \frac{1}{2}$$

So the limit exists and is $\frac{1}{2}$. See graph at the end.

(iii) $\lim_{(x,y)\to(0,0)} \frac{2x+y}{x+y} = \frac{0}{0}$

See the graphs. In particular the contour graph shows level curves that pass through (0,0). This indicates that the limit DNE. To show this look at paths of the form $y = mx$ where m is a parameter. Then the limit becomes

$$\lim_{x\to 0} \frac{2x+mx}{x+mx}$$

$$= \lim_{x\to 0} \frac{2+m}{1+m} = \frac{2+m}{1+m}.$$

This shows that for different values of m, we get different values for the limit. Thus, the limit DNE.

(iv) $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y} = \frac{0}{0}$

If we try $y=mx$, then $\lim_{x\to 0} \frac{x^2}{x^2+mx} = 0$.
This only tells us that IF the limit exists, THEN it is zero. But try $y = mx^2$, then

$$\lim_{x\to 0} \frac{x^2}{x^2+mx^2} = \frac{1}{1+m} \neq \text{constant } \forall m$$

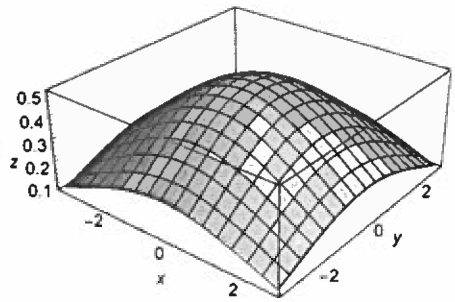
therefore the limit DNE. See graph.

For these indeterminant forms, we pick paths that "even out" the powers.

Example (ii)

In[2]:= Plot3D$\left[\frac{1 - \text{Cos}\left[\text{Sqrt}\left[x^2 + y^2\right]\right]}{x^2 + y^2}\right.$, {x, -3, 3}, {y, -3, 3}, PlotPoints → 50, AxesLabel → {x, y, z}$\left.\right]$

Out[2]=



Example (iii)

In[3]:= Plot3D$\left[\frac{2x + y}{x + y}\right.$, {x, -3, 3}, {y, -3, 3}, PlotPoints → 50, AxesLabel → {x, y, z}$\left.\right]$

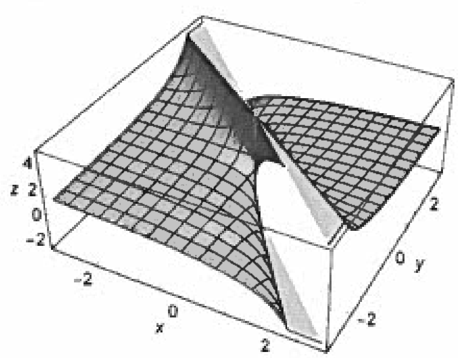
⋯ Power: Infinite expression $\frac{1}{0.}$ encountered.

⋯ Power: Infinite expression $\frac{1}{0.}$ encountered.

⋯ Power: Infinite expression $\frac{1}{0.}$ encountered.

⋯ General: Further output of Power::infy will be suppressed during this calculation.

Out[3]=



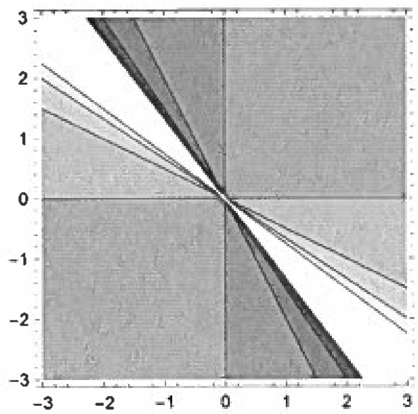
In[5]:= ContourPlot$\left[\frac{2x+y}{x+y}, \{x, -3, 3\}, \{y, -3, 3\}\right]$

... Power: Infinite expression $\frac{1}{0.}$ encountered.

... Power: Infinite expression $\frac{1}{0.}$ encountered.

... Power: Infinite expression $\frac{1}{0.}$ encountered.

... General: Further output of Power::infy will be suppressed during this calculation.

Out[5]=


Example (iv)

In[6]:= ContourPlot$\left[\frac{x^2}{x^2+y}, \{x, -3, 3\}, \{y, -3, 3\}\right]$

Out[6]=