§14.3 Partial Derivatives

Recall: $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$
provided the limit exists.

Definition Let f be a function of two variables. Then
$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$
and $$f_y(x,y) = \lim_{k \to 0} \frac{f(x,y+k) - f(x,y)}{k},$$
provided the limits exist.

Notation for the partial of f w.r.t x:
$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[f(x,y)] = D_x[f](x,y)$$

- This definition can be extended to functions of more than two variables.
- The dèfinition tells us how to "calculate" these definitions: for a partial w.r.t. x, think of y as being a constant. For a partial w.r.t. y, think of x as a constant.
- We interpret f_x as the "slope" of the surface in the x-direction, f_y is the slope in the y-direction.

Example $f(x,y) = x^2\cos(xy)$. Find f_x and f_y.

For f_x we have a product rule:

$$f_x(x,y) = 2x\cos(xy) - x^2y\sin(xy)$$

chain rule

For f_y we only have a chain rule

$$f_y(x,y) = -x^3\sin(xy)$$

Second order derivatives:

$$f_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right]$$

$$f_{xy}(x,y) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]$$

$$f_{yx}(x,y) = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]$$

$$f_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right]$$

Theorem (Clairaut's) suppose f is defined on a disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} are continuous on D, then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

Provided everything is "nice", for higher derivatives

$$\frac{\partial^{m+n} f}{\partial x^m \partial y^n}$$

are all the same, no matter the order in which we take the derivatives.

For Example: $f_{xxy} = f_{xyx} = f_{yxx} = \dfrac{\partial^3 f}{\partial x^2 \partial y}$

So, provided everything is continuous, there are

(a) two first derivatives,
(b) three second derivatives,
(c) four third derivatives,
$\vdots$

All of the properties of derivative from single variable calculus, still apply for partial derivatives.