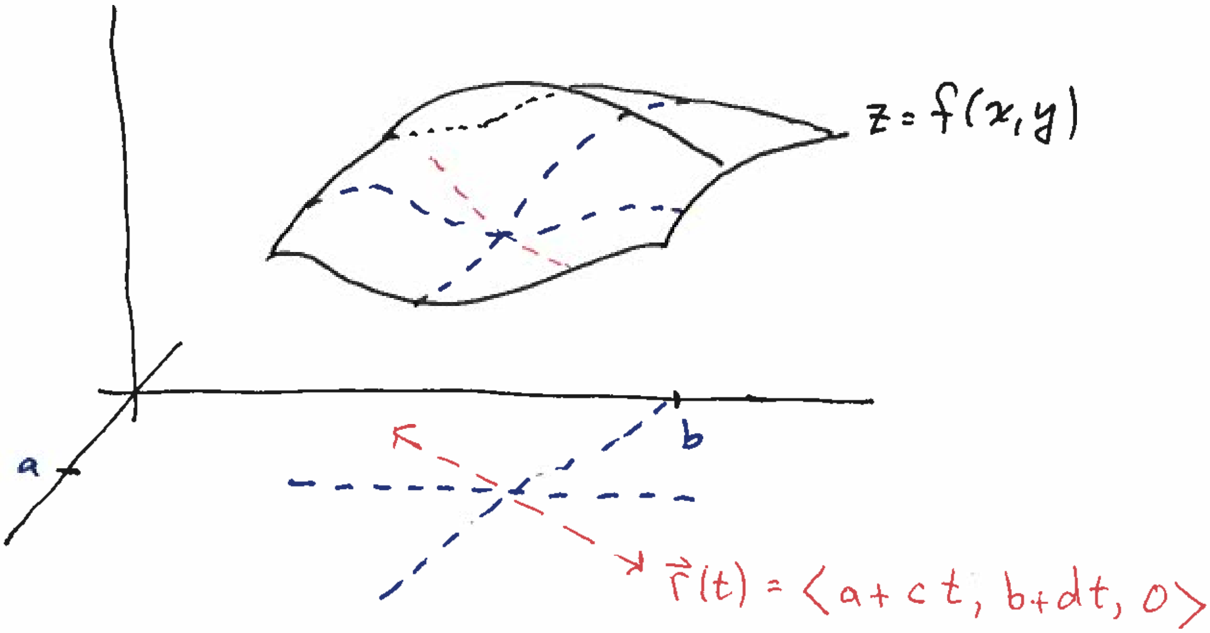
§14.6 Directional Derivatives and The Gradient Vector



Directional Derivatives

Let $\vec{u} = \langle c, d \rangle$ where $|\vec{u}| = 1$. Then the direction derivatives of f in the direction of $\vec{u}$ is given by

$$D_{\vec{u}}[f](a,b) = \lim_{\Delta t \to 0} \frac{f(\vec{r}(\Delta t)) - f(\vec{r}(0))}{\Delta t}$$

Note that if $|\vec{u}| \neq 1$, then the denominator would be $|\vec{u}|\Delta t$.

Also note that $D_{\vec{u}}[f](a,b)$ is just the chain rule:

$$D_u[f](a,b) = \frac{d}{dt}\left[f(x,y)\right] \quad \text{where } \begin{array}{l} x = a+ct \\ y = b+dt \end{array}$$
$$= \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$
$$= c\, f_x(a,b) + d\, f_y(a,b)$$
$$= \langle c, d\rangle \circ \langle f_x(a,b), f_y(a,b)\rangle$$
$$= \vec{u}\circ \underbrace{\nabla f(a,b)}_{\text{the gradient of } f \text{ at } (a,b)}$$

∇ is the gradient opperator: $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\rangle$.

__Example__ Let $f(x,y) = x^2\sin(y) + \cos(xy)$ and $\vec{v} = \langle 1,1\rangle$. Find $D_{\vec{v}}[f](1,\frac{\pi}{2})$.

First $|\vec{v}| = \sqrt{2}$ so let $\vec{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$ and then $D_{\vec{v}}[f](1,\frac{\pi}{2}) = D_{\vec{u}}[f](1,\frac{\pi}{2}) = \vec{u}\circ\nabla f(1,\frac{\pi}{2})$.

Next, $\nabla f = \langle 2x\sin(y) - y\sin(xy), x^2\cos(y) - x\sin(xy)\rangle$ so $\nabla f(1,\frac{\pi}{2}) = \langle 2-\frac{\pi}{2}, 0-1\rangle$ and

$$D_{\vec{u}}[f](1,\tfrac{\pi}{2}) = \langle \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\rangle \circ \langle 2-\tfrac{\pi}{2}, -1\rangle$$
$$= \frac{3-\pi}{2\sqrt{2}} \approx -0.0501$$

The Gradient Vector

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$

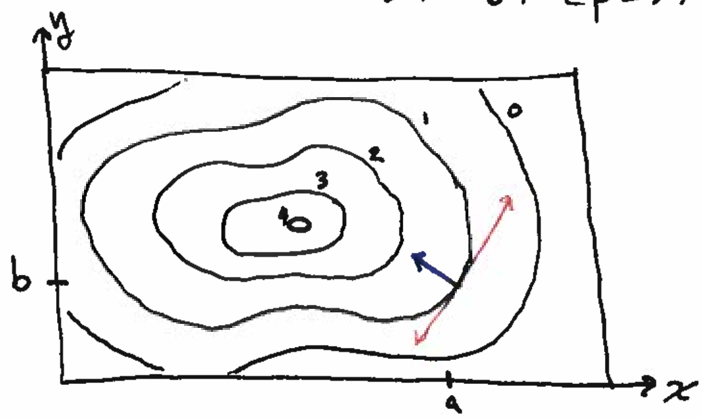
We juct saw that

$$D_{\vec{u}}[f](x,y) = \vec{u} \cdot \nabla f(x,y)$$

Since $\nabla f(a,b)$ is a "constant" vector and $|\vec{u}|=1$, the only part of this that can change is the direction of $\vec{u}$. That is

$$D_{\vec{u}}[f](a,b) = |u|\,|\nabla f(a,b)| \cos(\theta)$$
$$= |\nabla f(a,b)| \cos(\theta)$$

where θ is the angle between $\vec{u}$ and $\nabla f(a,b)$. Thus $D_{\vec{u}}[f](a,b) = 0$ when $\theta = \pm\frac{\pi}{2}$ and $D_{\vec{u}}[f](a,b)$ is largest when $\theta = 0$. So $\nabla f(a,b)$ is in the direction of steepest ascent.



Contour graph of $z = f(x,y)$

tangent line to a level curve $D_{\vec{u}}[f] = 0$

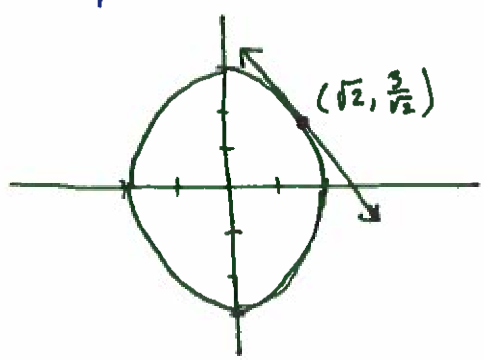
$\nabla f(a,b)$ is orthogonal to level curve.

So the equation of the tangent to the level curve is

$$\nabla f(a,b) \circ \langle x-a, y-b \rangle = 0$$

$$\Leftrightarrow \quad f_x(a,b)(x-a) + f_y(a,b)(y-b) = 0$$

<u>Example</u> Find the equation of the tangent line that passes through $(\sqrt{2}, \frac{3}{\sqrt{2}})$ on the curve $\frac{x^2}{4} + \frac{y^2}{9} = 1$.



The curve is a level curve of $f(x,y) = \frac{x^2}{4} + \frac{y^2}{9}$. So $\nabla f = \langle \frac{1}{2}x, \frac{2}{9}y \rangle$

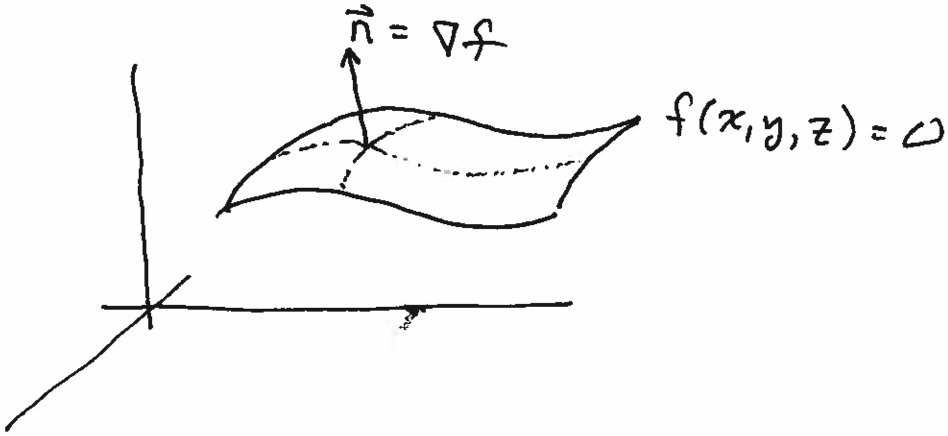
$$\nabla f(\sqrt{2}, \tfrac{3}{\sqrt{2}}) = \left\langle \frac{\sqrt{2}}{2}, \frac{2\sqrt{2}}{6} \right\rangle = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{3} \right\rangle$$

and the equation of the tangent is

$$\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{3} \right\rangle \circ \left\langle x-\sqrt{2}, y-\frac{3}{\sqrt{2}} \right\rangle = 0$$

$$\Leftrightarrow \quad \frac{\sqrt{2}}{2}(x-\sqrt{2}) + \frac{\sqrt{2}}{3}\left(y - \frac{3}{\sqrt{2}}\right) = 0$$

Extension to 3-D:



Tangent Planes:

$$\nabla f(x_0, y_0, z_0) \circ \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

Example Find equation of tangent plane through (2,3,1) on the surface

$$x^2 + 2y^2 - 3z^2 = 19.$$

This is a level surface of the function

$$F(x,y,z) = x^2 + 2y^2 - 3z^2$$

so $\vec{n} = \nabla F(2,3,1)$.

$$\nabla F = \langle 2x, 4y, -6z \rangle$$

$$\nabla F(2,3,1) = \langle 4, 12, -6 \rangle$$

Tangent plane:

$$\langle 4, 12, -6 \rangle \circ \langle x-2, y-3, z-1 \rangle = 0 \qquad \div 2 \text{ to get}$$

$$2(x-2) + 6(y-3) - 3(z-1) = 0$$

$$2x + 6y - 3z = 19$$