14.8 Lagrange Multipliers

Introduction

Consider the following problem: you are walking along a trail that is on a mountain. If the mountain can be described by the equation

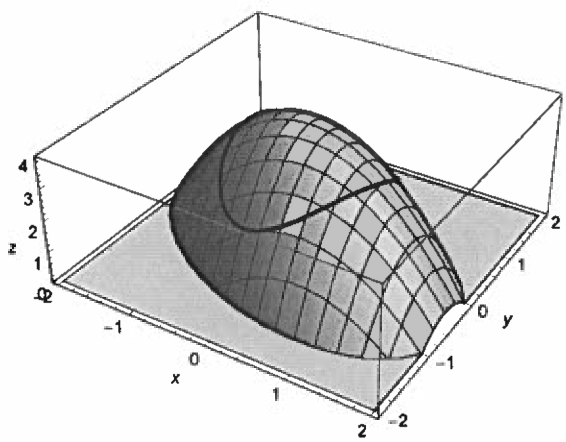
In[1]:=
```
f[x_, y_] = 4 - x^2 - x y - 2 y^2;
```

and the trail is the part of the mountain where $x^2 + y^2 = 1$, then at what (x, y) values on the trail are you at the highest (or lowest) altitude?

First let's get a picture of this situation. Note that the trail is the zero-level curve of $g(x, y) = x^2 + y^2 - 1$.

In[2]:=
```
g[x_, y_] = x^2 + y^2 - 1;
mountain = Plot3D[f[x, y], {x, -2, 2}, {y, -2, 2},
   PlotRange → {0, 4}, PlotPoints → 20, AxesLabel → {x, y, z}];
trail = ParametricPlot3D[{Cos[t], Sin[t], f[Cos[t], Sin[t]]},
   {t, 0, 2 Pi}, PlotStyle → Directive[Thick, Blue]];
Show[mountain, trail]
```

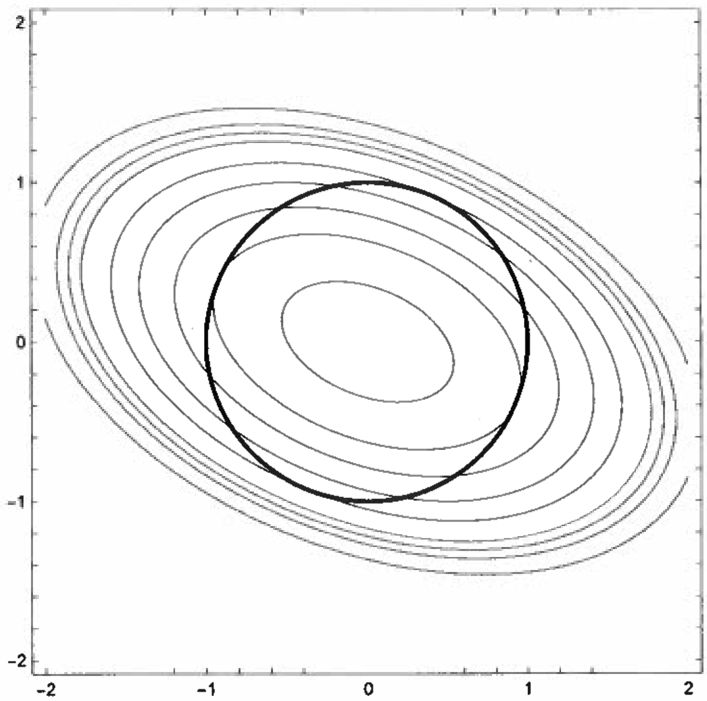
Out[5]=



Now look at the level curves for the mountain and a birds-eye view of the trail.

In[6]:=
```
levelmount = ContourPlot[f[x, y], {x, -2, 2},
   {y, -2, 2}, Contours → {.25, .75, 1, 1.25, 1.79, 2.25, 2.75, 3.2, 3.75},
   PlotPoints → 40, ContourShading → False];
leveltrail = ParametricPlot[{Cos[t], Sin[t]}, {t, 0, 2 Pi},
   PlotStyle → Directive[Thick, Blue]];
Show[levelmount, leveltrail]
```

Out[6]=



Now, it we start at (0, 1) and walk counter-clockwise, then we cross level curves of f which indicates that we are going uphill until we get to about (−.9, .4) where we are at a height of 3.2 (a high point on the trail). As we continue, we cross level curves of f until we reach a height of 1.79 at about (−.4, −.9). This is a low point on the trail. We then go up hill until we reach a high point at about (.9, −.4) and then downhill until we reach the other low point.

Notice that the trail (constraint) is tangent to a level curve of the mountain (suffice) at points that are extreme on the trail. We can find possible extreme of a function, subject to a constraint, by finding these points of tangency. We will use the gradient to do this. Since the gradient is orthogonal to the level curve, (a, b) is a critical point if

$$\nabla f(a, b) = \lambda \nabla g(a, b),$$

where λ is a constant. That is, the gradient of f is parallel to the gradient of g. In the above problem, this gives us two equations in three variables. The third equation is given by the constraint:

```
In[9]:= dfdx = D[f[x, y], x];
        dfdy = D[f[x, y], y];
        dgdx = D[g[x, y], x];
        dgdy = D[g[x, y], y];
        cp = NSolve[{dfdx == λ * dgdx, dfdy == λ * dgdy, g[x, y] == 0}, {x, y, λ}]

Out[13]= {{x → 0.382683, y → 0.92388, λ → -2.20711}, {x → -0.92388, y → 0.382683, λ → -0.792893},
          {x → -0.382683, y → -0.92388, λ → -2.20711}, {x → 0.92388, y → -0.382683, λ → -0.792893}}
```

To find the heights, we plug the critical points into the function describing the surface to get:

```
In[14]:= f[x, y] /. cp

Out[14]= {1.79289, 3.20711, 1.79289, 3.20711}
```

So the high point(s) have an altitude of about 3.2 at coordinates (–.92, .38) and (.92, –.38). The low points on the trail have an altitude of about 1.79 at coordinates (.38, .92) and (–.38, –.92).

More Examples:

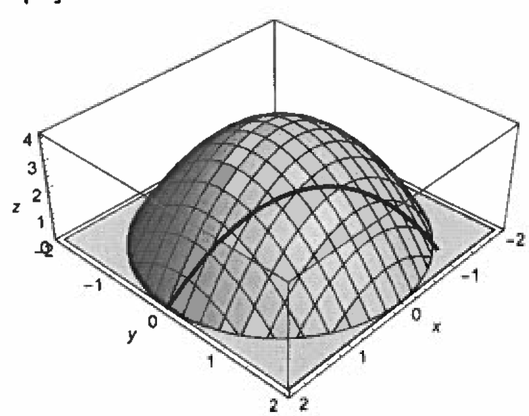
Example (i)

Find the max of $z = 4 - x^2 - y^2$, if $x + 2y = 2$.

Here is a graph. See hand written notes for solution.

```
In[15]:= f[x_, y_] = 4 - x^2 - y^2;
         p1 = Plot3D[f[x, y], {x, -2, 2}, {y, -2, 2},
            PlotRange → {0, 4}, PlotPoints → 20, AxesLabel → {x, y, z}];
         p2 = ParametricPlot3D[{2 - 2 t, t, f[2 - 2 t, t]}, {t, 0, 2},
            PlotStyle → Directive[Thick, Blue]];
         Show[
          p1,
          p2]
```

Out[18]=



$f(x,y) = 4 - x^2 - y^2$, constraint: $\underbrace{g(x,y) = x+2y}_{\text{set}} = 2$

$$\nabla f = \lambda \nabla g \Rightarrow \left.\begin{aligned} -2x &= (\lambda)(1) \\ -2y &= (\lambda)(2) \\ \text{constraint:} \quad x+2y &= 2 \end{aligned}\right\} \text{ solve system}$$

From the first eq. $\Rightarrow \lambda = -2x$

plug into second eq. $\Rightarrow -2y = -4x \Rightarrow y = 2x$

plug into constraint $\Rightarrow x + 4x = 2 \Rightarrow x = \frac{2}{5}$

So c.p. is $\left(\frac{2}{5}, \frac{4}{5}\right)$

From the geometry (graph), there is a maximum at $\left(\frac{2}{5}, \frac{4}{5}\right)$. $f\left(\frac{2}{5}, \frac{4}{5}\right) = 4 - \frac{4}{25} - \frac{16}{25} = \frac{16}{5}$, so the maximum value is $\frac{16}{5}$.

Example (ii) find location of extreme of $f(x,y,z) = 8x^2 + 4yz - 16z$, subject to $4x^2 + 4z^2 = 16 - y^2$.

Here $g(x,y,z) = 4x^2 + 4z^2 + y^2$

$$\nabla f = \lambda \nabla g \Rightarrow \left.\begin{aligned} 16x &= (\lambda)(8x) \\ 4z &= (\lambda)(2y) \\ 4y - 16 &= (\lambda)(8z) \\ \text{constraint} \Rightarrow \quad 4x^2 + y^2 + 4z^2 &= 16 \end{aligned}\right\} \text{ system to solve}$$

From the first equation

$\lambda = 2$

then the system is

$$\left.\begin{aligned} z &= y \\ y-4 &= 4z \\ 4x^2+y^2+4z^2 &= 16 \end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned} y-4 &= 4y \\ 4x^2+y^2+4y^2 &= 16 \end{aligned}\right\} \Rightarrow$$

$$y = -\frac{4}{3}$$

$$x = \pm\frac{4}{3}$$

$$z = -\frac{4}{3}$$

c.p. $\left(\pm\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\right)$

or

$x = 0$

then the system is

$$\left.\begin{aligned} 2z &= \lambda y \\ y-\cancel{z}4 &= 2\lambda z \\ \cancel{4x^2} + y^2 + 4z^2 &= 16 \end{aligned}\right\}$$

Note that the first eq $\Rightarrow$

$\lambda = \frac{2z}{y}$ or $\underbrace{y = z = 0}_{\text{can not happen}}$

So

$$\left.\begin{aligned} y-\cancel{z}4 &= 2\left(\frac{2z}{y}\right)z \\ y^2+4z^2 &= 16 \end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned} 4z^2 &= y^2-4y \\ y^2+4z^2 &= 16 \end{aligned}\right\} \Rightarrow$$

$$y^2-2y = 8 \Rightarrow$$

$$(y-4)(y+2) = 0$$

c.p. $(0,-2,\pm\sqrt{3})$

$(0,4,0)$

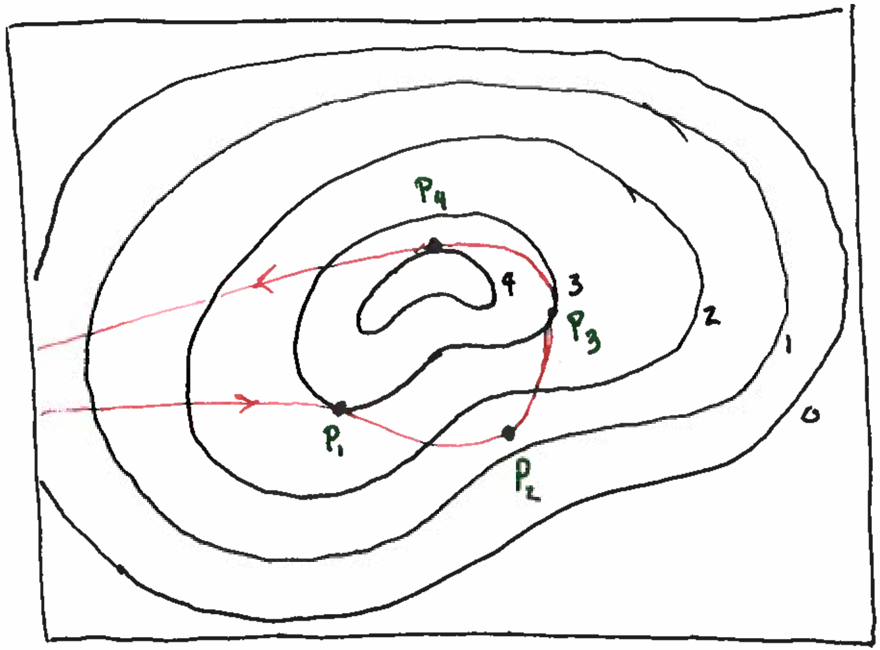
$$f\left(\pm\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\right) = \frac{128}{3} \approx 42.\overline{6}$$

$$f(0,-2,-\sqrt{3}) = 24\sqrt{3} \approx 41.57$$

$$f(0,-2,\sqrt{3}) = -24\sqrt{3} \approx -41.57$$

$$f(0,4,0) = 0$$

So we can say that there ~~is a~~ are maximums at $\left(\pm\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\right)$ an a minimum at $(0,-2,\sqrt{3})$. The other two points may be "saddles".



— level curves of $f(x,y)$

— constraint

- point P_1 is a local max. $f(P_1) = 3$
- point P_2 is a local min (Note that there is a level curve, not shown, of f that is tangent to the constraint at P_2). $1 < f(P_2) < 2$
- point P_3 is a point of inflection on the constraint. $f(P_3) = 3$
- point P_4 is a max. $f(P_4) = 4$

Note that at P_3 the level curve of f is tangent to the constraint (so $\nabla f(P_3) // \nabla g(P_3)$), but the constraint is increasing on both sides of P_3.

<u>Example (iii)</u> Find extreme of $f(x,y,z) = 3x - y - 3z$ subject to both $x + y - z = 0$ and $x^2 + 2z^2 = 1$.

Here we have two constraints:

$g(x,y,z) = x + y - z$ equals zero and

$h(x,y,z) = x^2 + 2z^2$ equals one.

The theory is similar, but we use two multipliers and

$$\nabla f = \lambda \nabla g + \mu \nabla h \implies$$

$$\left.\begin{aligned} 3 &= (\lambda)(1) + (\mu)(2x) \\ -1 &= (\lambda)(1) + (\mu)(0) \\ -3 &= (\lambda)(-1) + (\mu)(4z) \end{aligned}\right\} \text{3 eq. in 5 variables so add the 2 constraints}$$

$$x + y - z = 0$$
$$x^2 + 2z^2 = 1$$

Solve the system to get

c.p. $\left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{3}{2}}, \frac{1}{\sqrt{6}}\right)$ and $\left(\sqrt{\frac{2}{3}}, -\sqrt{\frac{3}{2}}, -\frac{1}{\sqrt{6}}\right)$

plug into $f \implies$ min max

Example (iv) Find the point on the curve $2x+3y=6$ that is closest to the origin.

Here we want to find the minimum distance from (x,y) to the $(0,0)$, subject to $2x+3y=6$. Minimizing distance is the same as minimizing the square of distance, so let

$$f(x,y) = x^2+y^2 \quad \text{which is } (\text{distance})^2$$

constraint: $g(x,y) = 2x+3y$ equals 6.

Then $\nabla f = \lambda \nabla g \Rightarrow$

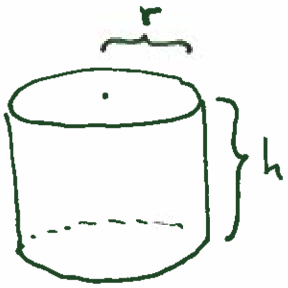
$$\left.\begin{aligned} 2x &= (\lambda)(2) \\ 2y &= (\lambda)(3) \\ 2x+3y &= 6 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} x &= \lambda \\ y &= \tfrac{3}{2}\lambda \\ 2x+3y &= 6 \end{aligned}\right\} \Rightarrow 2\lambda + \tfrac{9}{2}\lambda = 6$$

$$\Rightarrow \frac{13}{2}\lambda = 6 \Rightarrow \lambda = \frac{12}{13}$$

So c.p. $\left(\frac{12}{13}, \frac{18}{13}\right)$. By the geometry of the problem this is the location that minimizes the distance.

Example (v) Find the dimensions of the optimal tin can with volumn fixed at one unit^3.

By optimal, we mean minimize the surface area. So minimize



$$S(r,h) = 2\pi r^2 + 2\pi r h$$

subject to $V(r,h) = \pi r^2 h$ equals one.

$$\nabla S = \lambda \nabla V \implies$$

$$\left.\begin{aligned} 4\pi r^2 + 2\pi h &= (\lambda)(2\pi r h) \\ 2\pi r &= (\lambda)(\pi r^2) \\ \pi r^2 h &= 1 \end{aligned}\right\} \implies$$

The second equation gives

$r = 0$ or $\lambda = \frac{2}{r}$

$\implies$ contradiction with $\pi r^2 h = 1$

$$\implies \left.\begin{aligned} 2r^2 + h &= \left(\tfrac{2}{r}\right)(r h) \\ \pi r^2 h &= 1 \end{aligned}\right\} \implies$$

$$\left.\begin{aligned} 2r &= h \\ \pi r^2 h &= 1 \end{aligned}\right\}$$

So the diameter equals the height.

Question: Why then are many cans taller then they are wide? think about it.