§15.10 Change of Variables

Reminder: Read the book, do the homework as practice.

Recall: Simple substitution

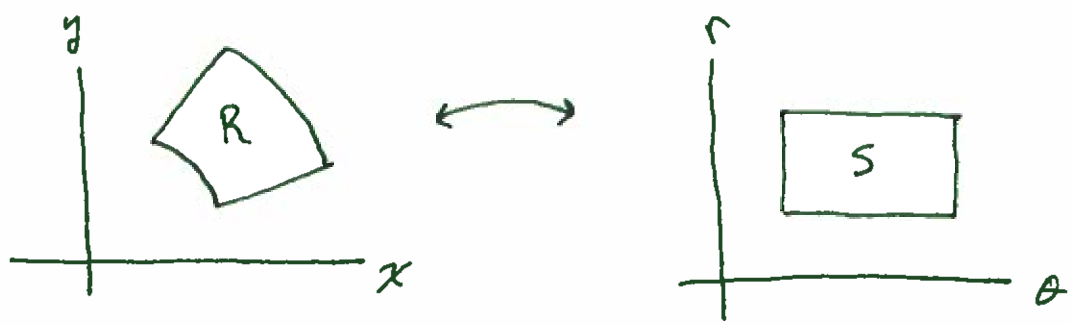
$$\int_a^b f(x)\,dx = \int_c^d f(g(u))\,g'(u)\,du$$

where $x = g(u)$, $dx = g'(u)du$, $a = g(c)$, $b = g(d)$

Polar:

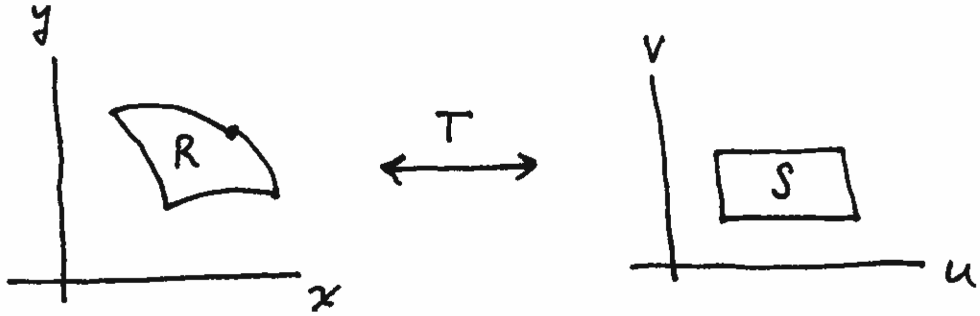
$$\iint_R f(x,y)\,dA = \iint_S f(r\cos(\theta), r\sin(\theta))\, r\,dr\,d\theta$$

Where $x = r\cos(\theta)$, $y = r\sin(\theta)$



The idea behind change of variables is to apply a transform, T, to the variables in the integral in order to simplify the integral. Either the region or the integrand.

If $T: \{x = g(u,v),\ y = h(u,v)\}$ gives



Let $\vec{r}(u,v) = g(u,v)\,\vec{i} + h(u,v)\,\vec{j}$

then

$$\vec{r}_u(u_0,v_0) = g_u(u_0,v_0)\,\vec{i} + h_u(u_0,v_0)\,\vec{j}$$
$$= \frac{\partial x}{\partial u}\,\vec{i} + \frac{\partial y}{\partial u}\,\vec{j}$$

Similarly

$$\vec{r}_v(u_0,v_0) = \frac{\partial x}{\partial v}\,\vec{i} + \frac{\partial y}{\partial v}\,\vec{j}$$

$R = T(S)$, that is the region R that maps to the region S through the transformation T can be approximate by the area of the parallelagram of the vectors

$$\vec{a} = \vec{r}(u_0+\Delta u, v_0) - \vec{r}(u_0,v_0) \approx \vec{r}_u\,\Delta u$$
$$\vec{b} = \vec{r}(u_0, v_0+\Delta v) - \vec{r}(u_0,v_0) \approx \vec{r}_v\,\Delta v$$

$$\text{area} = |\vec{a} \times \vec{b}| = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix} \Delta u \Delta v$$

Def If $(x,y) = T(u,v)$ is a transformation, then the Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix}$$

provided the partial derivatives exist.

For the transformation $(x,y) = T(u,v)$

$$dx\,dy = \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv$$

This is change of variables.

Examples

(i) Polar to Rectangular or visa-versa.

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

so

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{pmatrix}$$

$$= r\cos^2(\theta) + r\sin^2(\theta)$$

$$= r$$

thus $\quad dx\,dy = r\,dr\,d\theta$

Example

(ii) spherical $\leftrightarrow$ Rectangular (extension to 3D)

$$x = \rho \sin(\phi)\cos(\theta)$$
$$y = \rho \sin(\phi)\sin(\theta)$$
$$z = \rho \cos(\phi)$$

Note $\quad \dfrac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix}\nabla x & \nabla y\end{pmatrix}$

We extend this to 3D (or higher dimension)

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix}\nabla x & \nabla y & \nabla z\end{pmatrix}$$

where ∇x, ∇y and ∇z are collom vectors.

So $\quad \dfrac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \det\begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{pmatrix}$

$$= \det\begin{pmatrix} \sin(\phi)\cos(\theta) & \sin(\phi)\sin(\theta) & \cos(\phi) \\ \rho\cos(\phi)\cos(\theta) & \rho\cos(\phi)\sin(\theta) & -\rho\sin(\phi) \\ -\rho\sin(\phi)\sin(\theta) & \rho\sin(\phi)\cos(\theta) & 0 \end{pmatrix}$$

Expand this down the last collum :

$$= (-1)^{3+1}\cos(\varphi)\,\det\begin{pmatrix}\rho\cos(\varphi)\cos(\theta) & \rho\cos(\varphi)\sin(\theta)\\ -\rho\sin(\varphi)\sin(\theta) & \rho\sin(\varphi)\cos(\theta)\end{pmatrix}$$

$$+ (-1)^{3+2}(-\rho\sin(\varphi))\,\det\begin{pmatrix}\sin(\varphi)\cos(\theta) & \sin(\varphi)\sin(\theta)\\ -\rho\sin(\varphi)\sin(\theta) & \rho\sin(\varphi)\cos(\theta)\end{pmatrix}$$

$$+ (-1)^{3+3}(0)\,\det\begin{pmatrix}\sin(\varphi)\cos(\theta) & \sin(\varphi)\sin(\theta)\\ \rho\cos(\varphi)\cos(\theta) & \rho\cos(\varphi)\sin(\theta)\end{pmatrix}$$

$$= \cos(\varphi)\left[\rho^2\sin(\varphi)\cos(\varphi)\cos^2(\theta) + \rho^2\sin(\varphi)\cos(\varphi)\sin^2(\theta)\right]$$

$$+ \rho\sin(\varphi)\left[\rho\sin^2(\varphi)\cos^2(\theta) + \rho^2\sin^2(\varphi)\sin^2(\theta)\right]$$

Since $\cos^2(\theta)+\sin^2(\theta)=1$ we have

$$= \rho^2\sin(\varphi)\cos^2(\varphi) + \rho^2\sin^3(\varphi)$$

Factor out $\rho^2\sin(\varphi)$, then $\cos^2(\varphi)+\sin^2(\varphi)=1$ to get

$$= \rho^2\sin(\varphi)$$

Thus $$dx\,dy\,dz = \frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)}\,d\rho\,d\varphi\,d\theta$$

$$= \rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta$$

Example

(iii) Find the Jacobian of $x = u + v$, $y = u - v$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2$$

Note we could solve for u, v:

$$x + y = 2u \implies u = \tfrac{1}{2}x + \tfrac{1}{2}y$$
$$x - y = 2v \implies v = \tfrac{1}{2}x - \tfrac{1}{2}y$$

so

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = -\frac{1}{2}$$

This means:

$$dx\,dy = -2\,du\,dv \quad \text{or} \quad du\,dv = -\tfrac{1}{2}\,dx\,dy$$

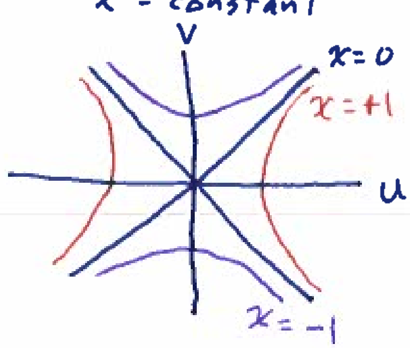
(iv) Find the Jacobian of $x = u^2 - v^2$, $y = u^2 + v^2$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 2u & 2u \\ -2v & 2v \end{pmatrix} = 8uv$$

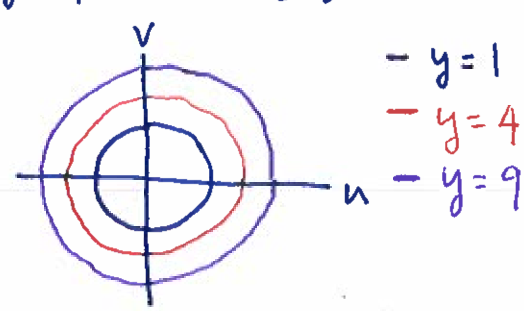
so $\quad dx\,dy = 8uv\,du\,dv$

Also:

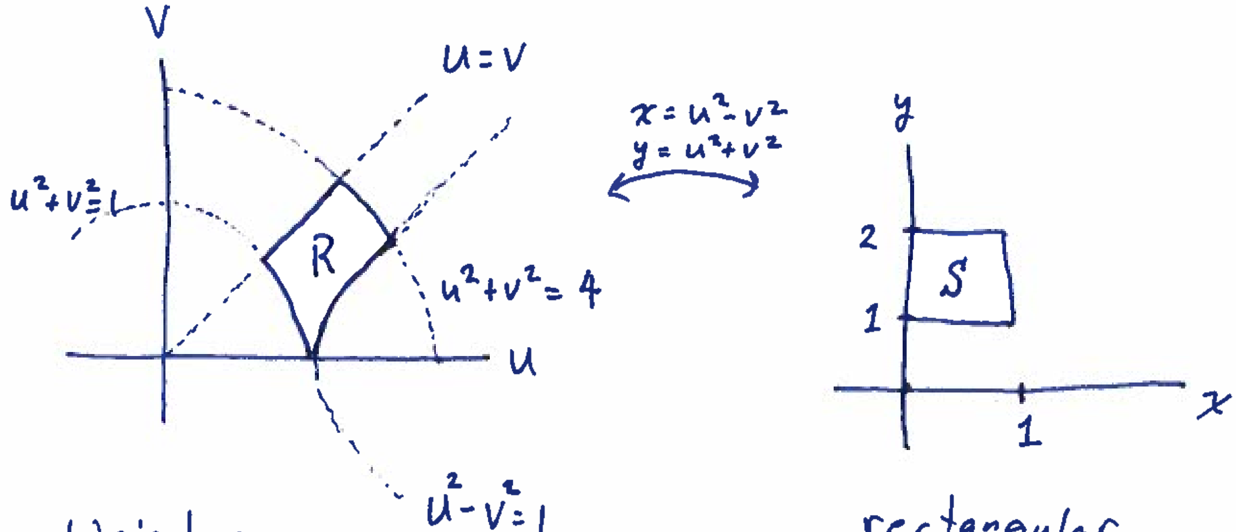
x = constant



y = positive constant



So



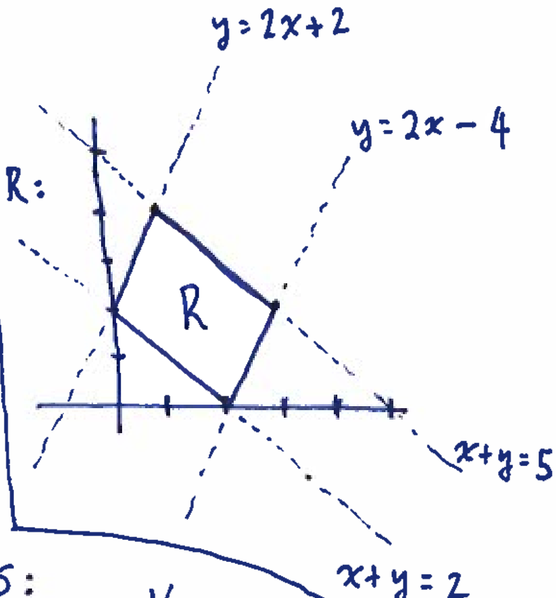
Weird region, not u-simple, nor v-simple

rectangular region

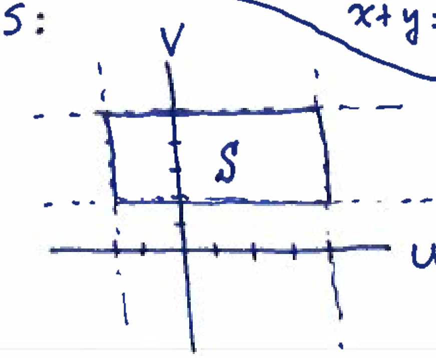
$u=v \iff x=0$
$u^2-v^2=1 \iff x=1$
$u^2+v^2=1 \iff y=1$
$u^2+v^2=4 \iff y=2$

Example

(v) Find $\iint_R (x+y)\,dx\,dy$ where R:



Note that the bounding curves are $x+y=2$, $x+y=5$, $2x-y=-2$ and $2x-y=4$. So let $u=2x-y$, $v=x+y$. then S:



$2x-y=-2 \iff u=-2$
$2x-y=4 \iff u=4$
$x+y=2 \iff v=2$
$x+y=5 \iff v=5$

$-2 \leq u \leq 4$
$2 \leq v \leq 5$

We need the Jacobian:

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} = 3$$

$$\Longleftrightarrow \quad du\,dv = 3\,dx\,dy \quad \text{or} \quad dx\,dy = \tfrac{1}{3}\,du\,dv$$

We need to convert the integrand:

$$u + v = 3x \Longleftrightarrow x = \tfrac{1}{3}u + \tfrac{1}{3}v$$

$$2v - \cancel{2}u = 3y \Longleftrightarrow y = \tfrac{2}{3}v - \tfrac{1}{3}u$$

So

$$x + y = v$$

Now we have

$$\iint_R (x+y)\,dx\,dy = \iint_S v \cdot \tfrac{1}{3}\,du\,dv$$

$$= \frac{1}{3}\int_2^5 \int_{-2}^4 v\,du\,dv$$

$$= \frac{1}{3}\int_2^5 6v\,dv$$

$$= 2\left(\tfrac{1}{2}v^2\right)\Big|_2^5$$

$$= 25 - 4$$

$$= 21$$

Final thoughts:

- From matrix algebra we have

$$\det(A) = \det(A^T)$$

So
$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} \nabla x & \nabla y & \nabla z \end{pmatrix}$$

where the gradients are callum vectors, can also be written as

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} \nabla x \\ \nabla y \\ \nabla z \end{pmatrix}$$

where the gradients are row vectors. The book uses the second form.

- I doesn't matter what variables you use. Remember that the variable of integration are "dummy" variables.

- A linear transformation $(x,y) = T(u,v)$ is one of the form

$$x = au + bv$$
$$y = cu + dv$$

where a, b, c and d are constants. In this case (but not generally)

$$\frac{\partial(x,y)}{\partial(u,v)} = ad - bc = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \text{constant}$$