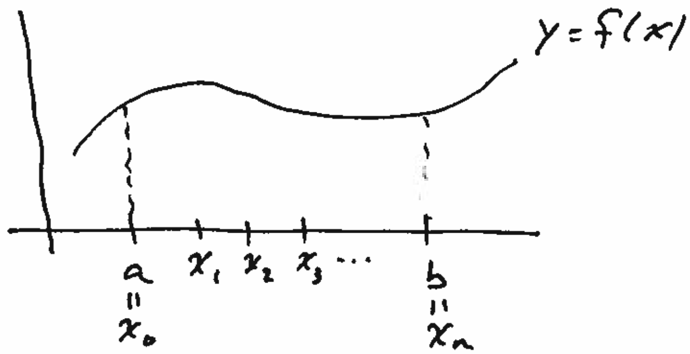
§15.1 Double Integrals Over Rectangular Regions

Recall from calc I:



$P = \{x_0, x_1, x_2, \ldots, x_n\}$ is a partition of $[a,b]$ where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. $\Delta x_i = x_i - x_{i-1}$ $\|P\| = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\}$ is the "size" of the partition. Finally $x_i^* \in [x_{i-1}, x_i]$ is an arbitrary value in the i^{th} subinterval. then

$$\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

provided the limit exists.

Theorem If f is continuous on $[a,b]$, then $\int_a^b f(x)\,dx$ exists. In this case we may take $\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$, x_i^* may be choosen in any way and

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

Now Calc III

see pictures on page 1001.

Let $R = [a,b] \times [c,d]$ be a rectangle in the xy-plane. If f is a continuous function on R, then

$$\iint_R f(x,y)\,dA = \lim_{(m,n)\to(\infty,\infty)} \sum_{k=1}^{m} \sum_{j=1}^{n} f(x_k^*, y_j^*)\Delta A$$

where $\Delta x = \frac{b-a}{m}$, $\Delta y = \frac{d-c}{n}$, $x_k = a + k\Delta x$, $y_j = c + j\Delta x$, $\Delta A = \Delta x \Delta y$. $x_k^* \in [x_{k-1}, x_k]$ and $y_j^* \in [y_{j-1}, y_j]$ can be choosen in any way.

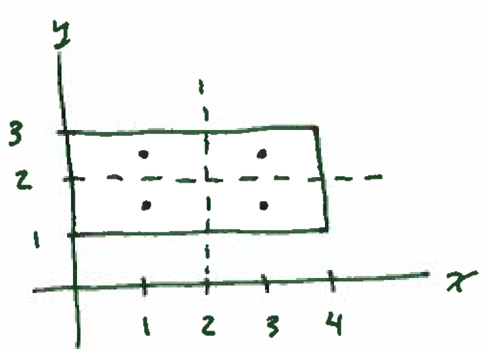
Midpoint Approximation:

x_k^* and y_j^* are choosen to be midpoints

Example Let $R = \{(x,y) : 0 \le x \le 4 \text{ and } 1 \le y \le 3\}$. Use the Midpoint Method to approximate

$$\iint_R f(x,y)\,dA$$

where $f(x,y) = x^2y - xy^2$. Use 2 subintervals in each direction.



$$\left.\begin{aligned} \Delta x &= \frac{4-0}{2} = 2 \\ \Delta y &= \frac{3-1}{2} = 1 \end{aligned}\right\} \Rightarrow \Delta A = 2$$

$x_1^* = 1$, $x_2^* = 3$

$y_1^* = 1.5$, $y_2^* = 2.5$

So $$M_{2,2} = \big(f(1,1.5) + f(1,2.5) + f(3,1.5) + f(3,2.5)\big)\Delta A$$

$$= \big((1.5 - 1.5^2) + (2.5 - 2.5^2) + (3^2(1.5) - 3(1.5^2)) + (3^2(2.5) - 3(2.5^2))\big)(2)$$

$$= 12$$

$\iint_R f(x,y)\,dA$ can be interpreted as the valumn under the surface $z = f(x,y)$ over the rectangle R.

<u>Definition</u> The average value of f over the rectangle $R = \{(x,y) : a \le x \le b, c \le y \le d\}$ is given by

$$\text{Avg}[f](R) = \frac{1}{(b-a)(d-c)} \iint_R f(x,y)\,dA.$$

Recall from calc I:

$$\text{Avg}[f]([a,b]) = \frac{1}{b-a}\int_a^b f(x)\,dx$$