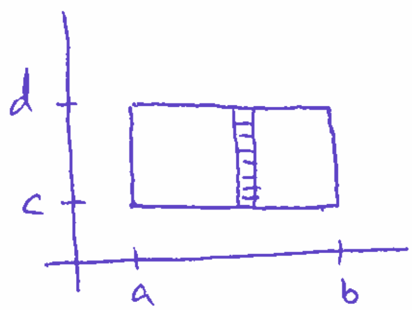
§15.2 Iterated Integrals

Let $R = [a,b] \times [c,d]$ be a rectangular region in the xy-plane. Then

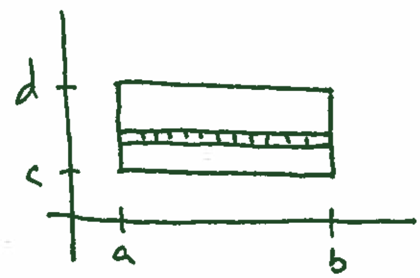
$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$
$$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$

$dy\,dx$



fix x, then integrate over y.

$dx\,dy$



fix y, then integrate over x.

Example (i)

$R = \{(x,y) : 0 \le x \le 4,\ 1 \le y \le 3\}$, find $\iint_R (x^2y - xy^2)\,dA$

We will do this using both orders of integration.

$dx\,dy$:

$$\iint_R (x^2y - xy^2)\,dA = \int_1^3 \underbrace{\int_0^4 (x^2y - xy^2)\,dx}_{\text{Do first thinking of } y \text{ as a constant}}\,dy$$

$$= \int_1^3 \left(\tfrac{1}{3}x^3y - \tfrac{1}{2}x^2y^2\right)\Big|_{x=0}^{x=4}\,dy$$

$$= \underbrace{\int_1^3 \left(\tfrac{64}{3}y - 8y^2\right)dy}_{\text{calc I integral}}$$

$$= \left(\tfrac{32}{3}y^2 - \tfrac{8}{3}y^3\right)\Big|_1^3$$

$$= (96 - 72) - \left(\tfrac{32}{3} - \tfrac{8}{3}\right)$$

$$= 16$$

$dy\,dx$:

$$\iint_R (x^2y - xy^2)\,dA = \int_0^4 \underbrace{\int_1^3 (x^2y - xy^2)\,dy}_{\text{Do first thinking of } x \text{ as a constant}}\,dx$$

$$= \int_0^4 \left(\tfrac{1}{2}x^2y^2 - \tfrac{1}{3}xy^3\right)\Big|_{y=1}^{y=3}\,dx$$

$$= \underbrace{\int_0^4 \left[\left(\tfrac{9}{2}x^2 - 9x\right) - \left(\tfrac{1}{2}x^2 - \tfrac{1}{3}x\right)\right]dx}_{\text{calc I integral}}$$

$$= 16$$

Example (ii) find $\int_{-1}^{1}\int_{0}^{1} x e^{xy}\, dx\, dy$

Note that, in this order of integration, we would need to do integration by parts for the first integral. If we change the order we get

$$\int_{0}^{1}\int_{-1}^{1} x e^{xy}\, dy\, dx.$$

Now the first (inner) integral is a simple substitution: $u = xy$, $du = x\,dy$. The integral becomes

$$\int_{0}^{1}\int_{y=-1}^{y=1} e^{u}\, du\, dy = \int_{0}^{1} e^{u}\Big|_{y=-1}^{y=1} dx$$

$$= \int_{0}^{1} e^{x} - e^{-x}\, dx$$

$$= e^{x} + e^{-x}\Big|_{0}^{1}$$

$$= e + e^{-1} - 2$$

The primary goal when preforming itterated integrals is to choose an order that makes the first (inner) integral as simple as possible.