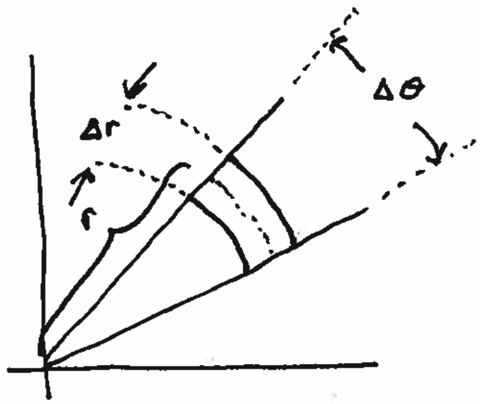
§15.4 Double Integrals in Polar



$$\Delta A = \frac{1}{2}\left(r+\frac{\Delta r}{2}\right)^2 \Delta\theta - \frac{1}{2}\left(r-\frac{\Delta r}{2}\right)^2 \Delta\theta$$

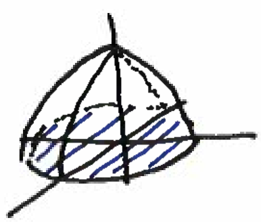
$$= \frac{\Delta\theta}{2}\left[\left(r^2 + r\Delta r + \frac{\Delta r^2}{4}\right) - \left(r^2 - r\Delta r + \frac{\Delta r^2}{4}\right)\right]$$

$$= \frac{\Delta\theta}{2}\left[2r\Delta r\right]$$

$$= r\,\Delta r\,\Delta\theta$$

Examples:

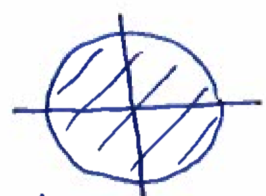
(i) Volume of a Sphere: top half is $z = \sqrt{R^2 - x^2 - y^2}$



Shadow region:

$x^2 + y^2 \leq R^2$

Use symmetry and look at first quadrant



$$V = 8\int_0^R \int_0^{\sqrt{R^2-x^2}} \sqrt{R^2-x^2-y^2}\,dy\,dx \quad \text{in rectangular}$$

$$= 8\int_0^{\frac{\pi}{2}} \int_0^R \sqrt{R^2-r^2}\cdot r\,dr\,d\theta \quad \text{in polar}$$

$\vdots$ simple sub. $u = R^2 - r^2$, $du = -2r\,dr$

$$= \frac{4}{3}\pi R^3$$

Recall the change of variable formulas:

Rec.	Polar	
x	$r\cos(\theta)$	
y	$r\sin(\theta)$	
$\pm\sqrt{x^2+y^2}$	r	$\Leftrightarrow \; r^2 = x^2+y^2$
$\tan^{-1}\left(\frac{y}{x}\right)$	θ	$\Leftrightarrow \; \theta = \cot^{-1}\left(\frac{x}{y}\right)$

Example

(ii) $I = \int_0^\infty e^{-x^2}\,dx$

$$I^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right)$$

Region of Integration

$$= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

$$= \int_0^{\pi/2} \int_0^\infty e^{-r^2} \cdot r\,dr\,d\theta \quad \text{in polar}$$

$$= \frac{\pi}{2} \int_0^{-\infty} -\frac{1}{2} e^u\,du$$

where $u = -r^2$
$du = -2r\,dr$

$$= \frac{\pi}{4}\, e^u \Big|_{-\infty}^{0}$$

$$= \frac{\pi}{4}$$

So $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$