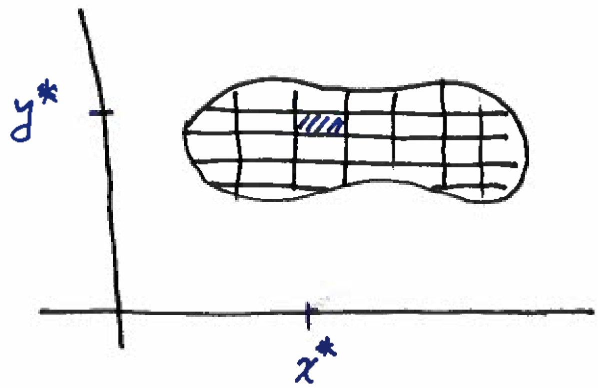
§15.5 Applications of Double Integrals

Center of Mass



little piece of mass at (x^*, y^*)

Small piece of mass = (density)(small area)

$$dm = \rho(x,y)\, dA$$

first moments = (mass)(distance)

about x-axis:

$$dM_x = y\, dm$$

about y-axis:

$$dM_y = x\, dm$$

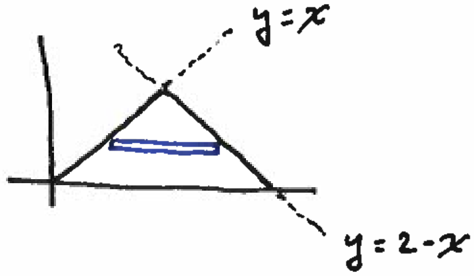
Integrate to get m, M_x, M_y and then the center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$$

Note that centroid is when $\rho(x,y) = 1$

Examples Find the center of mass

(i)



bounded by x-axis, $y=x$ and $y=2-x$, with $\rho(x,y)=xy$

This region is y-simple, but not x-simple. The intersection of $y=x$ and $y=2-x$ is $(1,1)$. So

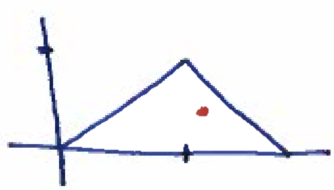
$$dm = xy\,dx\,dy$$

$$m = \int_0^1 \int_y^{2-y} xy\,dx\,dy \overset{CAS}{=} \frac{1}{3}$$

$$M_x = \int_0^1 \int_y^{2-y} xy^2\,dx\,dy \overset{CAS}{=} \frac{1}{6}$$

$$M_y = \int_0^1 \int_y^{2-y} x^2y\,dx\,dy \overset{CAS}{=} \frac{11}{60}$$

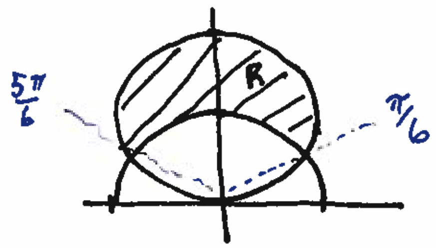
So $\bar{x} = \frac{M_y}{m} \overset{CAS}{=} \frac{11}{10} = 1.1$, $\bar{y} = \frac{M_x}{m} = \frac{1}{2} = .5$



seams reasonable

Example Find the centroid

(ii)



Inside $x^2+(y-1)^2=1 \Leftrightarrow$ In polar $r=2\sin(\theta)$

Outside $x^2+y^2=1 \Leftrightarrow r=1$

$$\Downarrow$$

$$1=2\sin(\theta)$$

$$\sin(\theta)=\frac{1}{2}$$

$$\theta=\frac{\pi}{6},\frac{5\pi}{6}$$

In Polar

$$m=\iint_R dA = \int_{\pi/6}^{5\pi/6}\int_1^{2\sin(\theta)} r\,dr\,d\theta$$

$$M_x=\iint_R y\,dA = \int_{\pi/6}^{5\pi/6}\int_1^{2\sin(\theta)} r\sin(\theta)\cdot r\,dr\,d\theta$$

$$M_y=\iint_R x\,dA = \int_{\pi/6}^{5\pi/6}\int_1^{2\sin(\theta)} r\cos(\theta)\cdot r\,dr\,d\theta$$

Using Mathematica we get

$$m=\frac{3\sqrt{3}+2\pi}{6},\quad M_x=\frac{3\sqrt{3}+8\pi}{12},\quad M_y=0$$

Note $M_y=0$ by symmetry could have been determined by the picture.

So

$$\bar{x}=0,\quad \bar{y}\approx 1.321$$

which seams reasonable.

Moment of Inertia (Second Moments)

This is a concept that is used in angular "stuff"

Moment of Inertia = (mass)(distance squared)

2nd Moment about x-axis:

$$I_x = \iint_R y^2 \, dm$$

2nd Moment about y-axis:

$$I_y = \iint_R x^2 \, dm$$

Most used: 2nd Moment about the origin

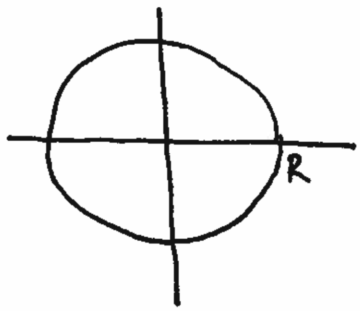
$$I_0 = \iint_R (x^2+y^2) \, dm$$

Radius of gyration about an axis (point)

$$mR^2 = I \Leftrightarrow R = \sqrt{\frac{I}{m}}$$

<u>Example</u> Radius of gyration of a solid wheel.

(iii) Consider the solid disk of density 1 and radius R.



the total mass = area

$$m = \pi R^2$$

the 2^{nd} moment about the origin in polar coordinates is

$$I_o = \iint_R (x^2+y^2)\,dm = \int_0^{2\pi}\int_0^R r^2 \cdot r\,dr\,d\theta$$

$$= 2\pi\left(\frac{1}{4}r^4\Big|_0^R\right)$$

$$= \frac{\pi R^4}{2}$$

So the radius of gyration about the origin is

$$\text{R.O.G.} = \sqrt{\frac{\pi R^4}{2} \div \pi R^2} = \frac{R}{\sqrt{2}}$$

That is, the disk has the same angular physics behaivior as a "perfect" bicycle wheel of radiu $\frac{R}{\sqrt{2}}$ and the same total mass.