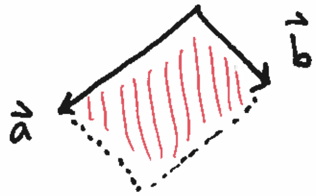
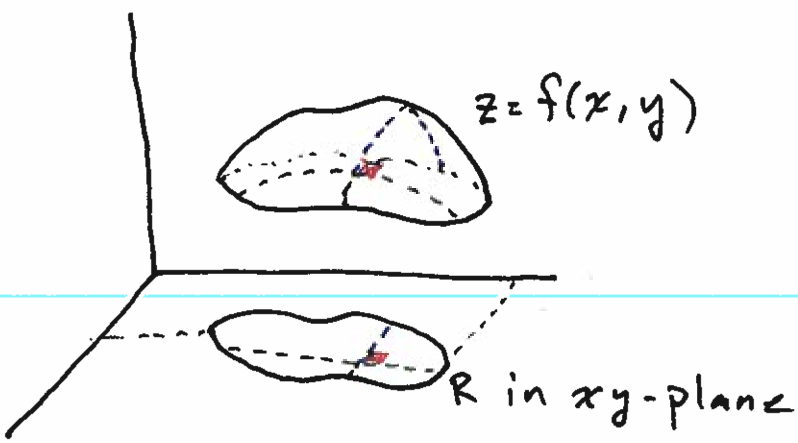
§15.6 Surface Area



$$\text{Area} = |\vec{a} \times \vec{b}|$$

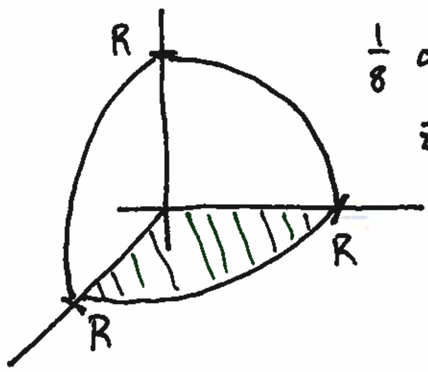
$$\vec{a} = \langle \Delta x, 0, f_x(x_0, y_0)\Delta x \rangle$$
$$\vec{b} = \langle 0, \Delta y, f_y(x_0, y_0)\Delta y \rangle$$
$$|\vec{a} \times \vec{b}| = \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\ dA$$



$$\text{Area of surface} = \iint_R \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\ dA$$

Examples

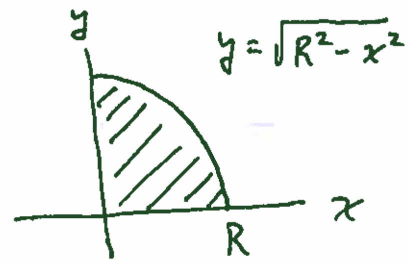
(i) Area of a sphere of radius R (surface Area)



$\frac{1}{8}$ of the sphere

$z = \sqrt{R^2 - x^2 - y^2}$

Shadow Region



little piece of area of surface:

$$dS = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\; dA$$

where

$$\frac{\partial f}{\partial x} = \frac{1}{2}\left(R^2 - x^2 - y^2\right)^{-1/2}(-2x) = \frac{-x}{\sqrt{R^2 - x^2 - y^2}}$$

$$\frac{\partial f}{\partial y} = \frac{-y}{\sqrt{R^2 - x^2 - y^2}} \quad \text{b/c of } xy \text{ symmetry}$$

So

$$dS = \sqrt{\left(\frac{-x}{\sqrt{R^2 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{R^2 - x^2 - y^2}}\right)^2 + 1}\; dA$$

$$= \sqrt{\frac{x^2}{R^2 - x^2 - y^2} + \frac{y^2}{R^2 - x^2 - y^2} + 1}\; dA$$

$$= \sqrt{\frac{R^2}{R^2 - x^2 - y^2}}\; dA$$

Since the region of integration and dS are easier in Polar we have

$$S \overset{*}{=} 8 \int_0^{\pi/2} \int_0^R \frac{R}{\sqrt{R^2 - r^2}} \, r\,dr\,d\theta \qquad u = R^2 - r^2, \quad du = -2r\,dr$$

$$\overset{\maltese}{=} -4R\left(\frac{\pi}{2}\right) \int_{R^2}^{0} \frac{1}{\sqrt{u}}\,du$$

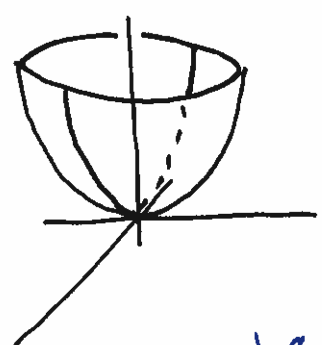
$$= 2\pi R \cdot 2u^{1/2} \Big|_0^{R^2}$$

$$= 4\pi R^2$$

* Note that the integral on the inside has no θ's and is a simple substitution.

✠ The minus sign will be used to flip the order of the limits.

(ii) $z = x^2 + y^2$ above $x^2 + y^2 \leq 4$



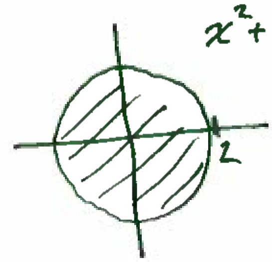
$z = x^2 + y^2$

$\frac{\partial z}{\partial x} = 2x$

$\frac{\partial z}{\partial y} = 2y$

Region of Integration

R:



$$dS = \sqrt{(2x)^2 + (2y)^2 + 1}\; dA$$

So Polar again:

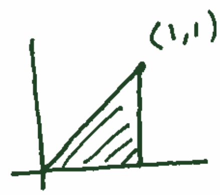
$$S \overset{*}{=} \int_0^{2\pi} \int_0^2 \sqrt{4r^2+1} \cdot r \, dr \, d\theta \qquad \begin{array}{l} u = 4r^2+1 \\ du = 8r\,dr \end{array}$$

$$= (2\pi)\left(\tfrac{1}{8}\right) \int_1^{17} \sqrt{u}\, du$$

$$= \left(\frac{\pi}{4}\right)\left(\frac{2}{3} u^{3/2}\right)\Big|_1^{17}$$

$$= \frac{\pi}{6}\left(17^{3/2} - 1\right) \approx 36.1769$$

(iii) $z = xy$ above



$$dS = \sqrt{(y)^2 + (x)^2 + 1} \; dA$$

Still might be easier in polar:

$$dS = \sqrt{r^2+1} \; r \, dr \, d\theta$$

Region: $y = x \Leftrightarrow \theta = \frac{\pi}{4}$

$x = 1 \Leftrightarrow r\cos(\theta) = 1 \Leftrightarrow r = \sec(\theta)$

So

$$S = \int_0^{\pi/4} \int_0^{\sec(\theta)} \sqrt{r^2+1} \; r \, dr \, d\theta \qquad \begin{array}{l} u = r^2+1 \\ du = 2r\,dr \end{array}$$

$$= \frac{1}{2} \int_0^{\pi/4} \int_1^{\sec^2(\theta)+1} \sqrt{u} \, du \, d\theta$$

$$S = \frac{1}{2}\int_0^{\pi/4} \left(\frac{2}{3}u^{3/2}\right)\Bigg|_1^{\sec^2(\theta)+1} d\theta$$

$$= \frac{1}{3}\int_0^{\pi/4} \left(\left(\sec^2(\theta)+1\right)^{3/2} - 1\right) d\theta$$

At this point do a numerical approximation

$$S = 0.640395$$

Note: While doing a numerical approximation of a double Integral is possible, it is often quicker to attempt to do one of the integrals by hand first.

Note: Much like arclength in Calc II, surface area often leads to integrals that must be approximated. So make sure you know how to do that with technology (calculator or Mathematica).