§16.1 Vector Fields

Recall: 1 variable $2x = 4 \Rightarrow x = 2$

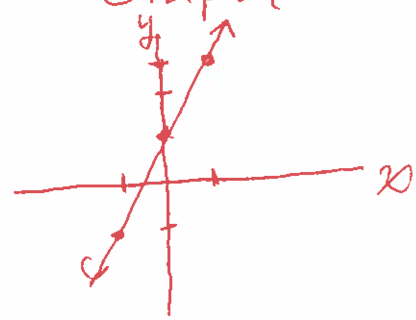
2 variables: one dependent, one independent

$y = 2x + 1$

Table

x	y
-1	-1
0	1
1	3

Graph



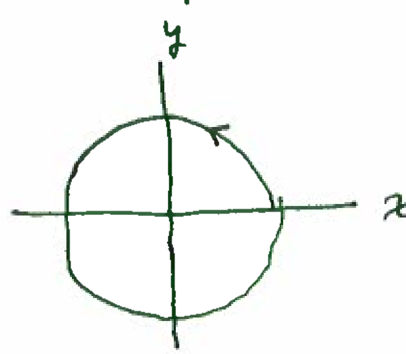
3 variables - parametric

$x(t) = \cos(t)$
$y(t) = \sin(t)$

Table

t	x	y
0	1	0
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{2}$	0	1

Graph



t does not appear in the graph and is called a parameter

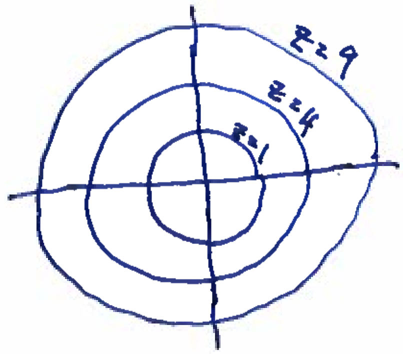
3 variables - two independent
on dependent

$$z = x^2 + y^2$$

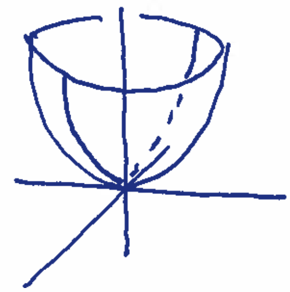
Table

x	y	z
0	0	0
0	1	1
1	0	1
1	1	2
2	0	4
⋮	⋮	⋮

Contour Graphs



Surface Graph



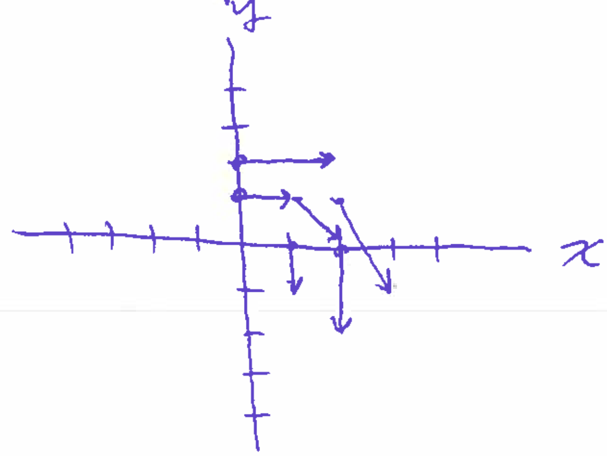
Next: vector fields: more than one independent variable and more than one dependent variable

$$\vec{F}(x,y) = \langle y, -x \rangle \iff \begin{cases} U(x,y) = y \\ V(x,y) = -x \end{cases}$$

Table

x	y	$\vec{F}$
0	1	$\langle 1, 0 \rangle$
0	2	$\langle 2, 0 \rangle$
1	0	$\langle 0, -1 \rangle$
2	0	$\langle 0, -2 \rangle$
1	1	$\langle 1, -1 \rangle$
2	1	$\langle 1, -2 \rangle$

Graph



If $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ ($\vec{F}$ maps 2D into 2D) then to "draw" the vector field: pick an (x_0, y_0), then draw the vector $\vec{F}(x_0, y_0)$ with the tail at (x_0, y_0).

This process is best left to a computer. Mathematica draws vector fields, but scales the vectors so the largest one in the window is a fixed "unit" length. So realize that Mathematica gives the correct direction, but not right scale. Examples to follow.

Gradient fields

$\nabla f(x,y)$ is a 2×2 vector field

Example

$$f(x,y) = x^2 + 2xy + y^3$$

$$\nabla f(x,y) = \langle 2x+2y, 2x+3y^2 \rangle \overset{set}{=} \langle P, Q \rangle$$

Notice that $P_y(x,y) = 2 = Q_x(x,y)$

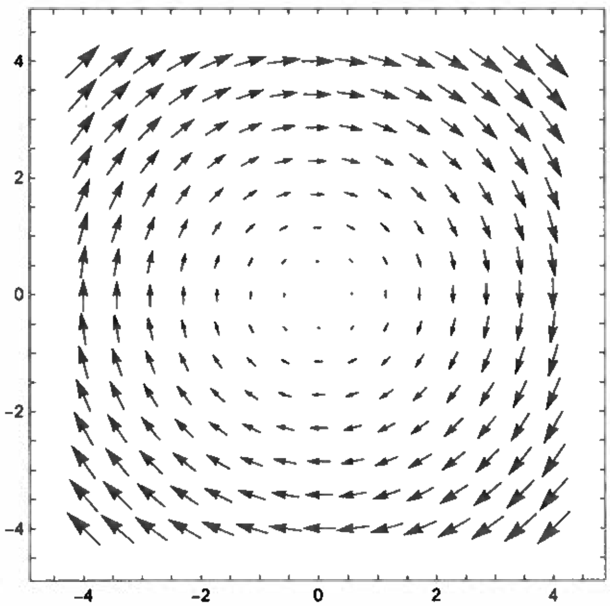
this always happens since $f_{xy} = f_{yx}$

16.1 Vector fields

For the vector field $\vec{F}(x, y) = \langle y, -x \rangle$, the graph looks like

In[2]:=
```
VectorPlot[{y, -x}, {x, -4, 4}, {y, -4, 4}]
```

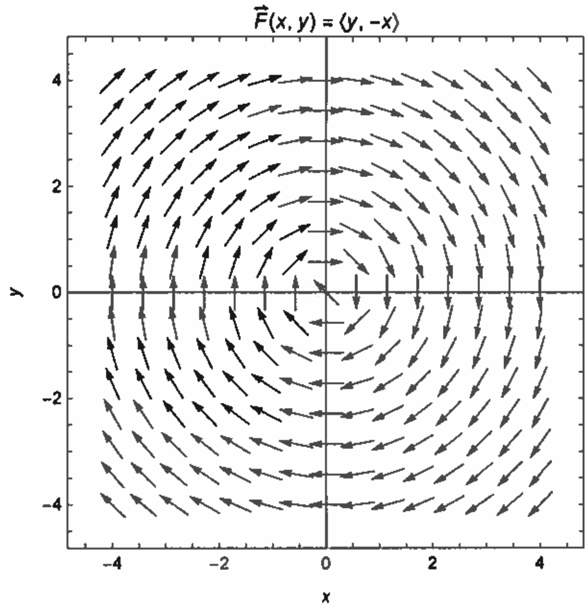
Out[2]=



This is the basic command. Notice that the largest vectors are near the corners of the window and are not the "proper" length. Here is the same vector field with some options to make all vectors the same length (and some other stuff to make the graph pretty).

In[1]:=
```
VectorPlot[
  {y, -x}, {x, -4, 4}, {y, -4, 4},
  FrameLabel → {x, y},
  Axes → True,
  VectorScale → {Small, Small, None},
  PlotLabel → "F⃗(x, y) = ⟨y, -x⟩"
]
```

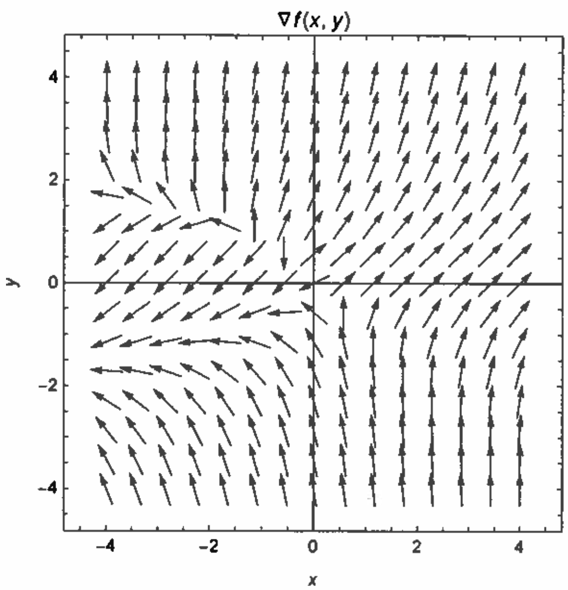
Out[1]=



It's a swirly! Here is the gradient field $\nabla f(x, y)$, where $f(x, y) = x^2 + 2xy + y^3$.

In[3]:=
```
VectorPlot[
 {2 x + 2 y, 2 x + 3 y^2}, {x, -4, 4}, {y, -4, 4},
 FrameLabel → {x, y},
 Axes → True,
 VectorScale → {Small, Small, None},
 PlotLabel → "∇f(x, y)"
]
```

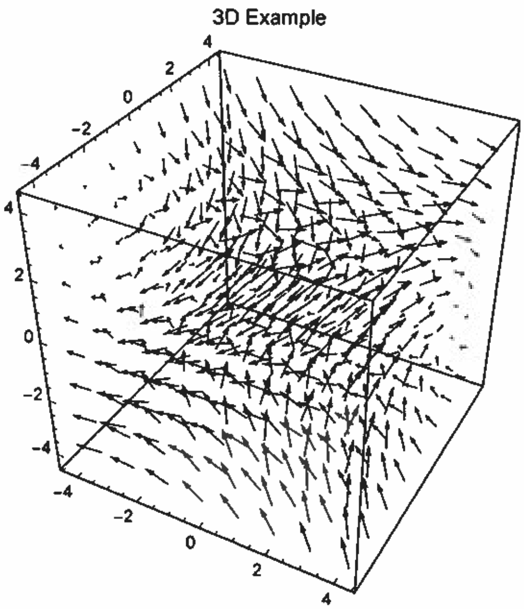
Out[3]=



Here is a 3D example:

In[4]:=
```
VectorPlot3D[
  {z, x, -y}, {x, -4, 4}, {y, -4, 4}, {z, -4, 4},
  Axes → True,
  VectorScale → {Small, Small, None},
  PlotLabel → "3D Example"
 ]
```

Out[4]=



Would you like to try to do that by hand? To really get the best idea of the "flow" of this vector field, you would need to "grab" the graph in Mathematica and rotate it to see it from several angles.

Then main use for vector field is to model fluid flow. This includes hydrology, meteorology, ocean engineering, ...