§16.2 Line Integrals

We have integrated over:

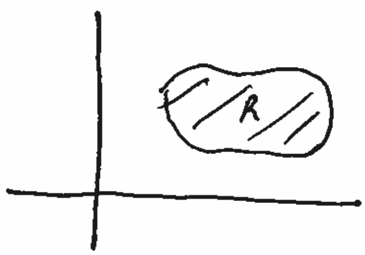
Intervals $a \le x \le b$

$$\int_a^b f(x)\,dx \overset{\text{def}}{=} \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta x_k$$

$$= F(b) - F(a)$$

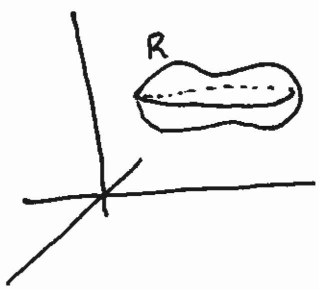
Fundemental theorem of Integrals

Regions in 2D



$$\iint_R f(x,y)\,dA$$

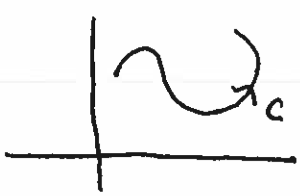
Regions in 3D



$$\iiint_R f(x,y,z)\,dV$$

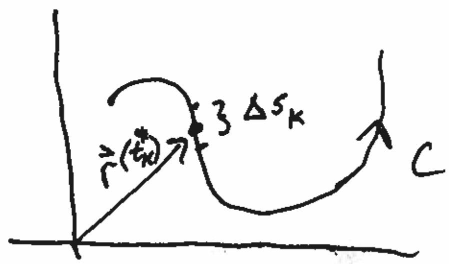
Next we will integrate along curves.

In 2D:



Let $C: \vec{r}(t) = \langle x(t), y(t) \rangle$ be a piecewise smooth curve in $\mathbb{R}^2$. Then

$$\int_C f(x,y)\,ds = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(\vec{r}(t_k^*))\,\Delta s_k$$



In practice we "plug in" the $\vec{r}(t)$:

Recall $\qquad ds = |\vec{r}\,'(t)|\,dt$

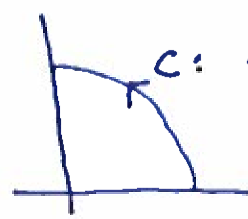
so if $C : \vec{r}(t)$ with $a \le t \le b$, then

$$\int_C f(x,y)\,ds = \int_a^b f(\vec{r}(t))\cdot|\vec{r}\,'(t)|\,dt$$

Example

(i)

$$I = \int_C (2 - xy)\,ds \qquad \text{where}$$



$C:\ x^2 + y^2 = 4$
$0 \le x \le 2$
oriented as shown

$$C : \vec{r}(t) = \langle 2\cos(t), 2\sin(t) \rangle$$
$$0 \le t \le \frac{\pi}{2}$$

so

$$I = \int_0^{\pi/2} \big(2 - (2\cos(t))(2\sin(t))\big)\sqrt{(-2\sin(t))^2 + (2\cos(t))^2}\,d$$

$$= \int_0^{\pi/2} (2 - 2\sin(2t))\sqrt{4}\,dt$$

$$= 4\left(t + \tfrac{1}{2}\cos(2t)\right)\Big|_0^{\pi/2}$$

$$= 4\left[\left(\frac{\pi}{2} + \frac{1}{2}(-1)\right) - \left(0 + \frac{1}{2}(1)\right)\right]$$

$$= 4\left(\frac{\pi}{2} - 1\right)$$

Line integrals with vector fields:

If $\vec{F}$ is a vector field, then

$$\int_C \vec{F}(x,y) \circ d\vec{r} = \int_C \vec{F}(x,y) \circ \vec{r}\,'(t)\,dt$$

<u>Example</u>

(ii) Let $C: \vec{r}(t) = \langle t, \cos(t) \rangle$ and $\vec{F}(x,y) = \langle -y, x \rangle$
$0 \le t \le \pi$

then

$$\int_C \vec{F}(x,y) \circ d\vec{r} = \int_0^\pi \langle -\cos(t), t \rangle \circ \langle 1, -\sin(t) \rangle\, dt$$

$$= \int_0^\pi (-\cos(t) - t\sin(t))\,dt$$

$\vdots$

Recall that $\vec{T} = \frac{\vec{r}\,'}{|\vec{r}\,'|}$ so $\vec{T}\,ds = \frac{\vec{r}\,'}{|\vec{r}\,'|} \cdot |\vec{r}\,'|\,dt = \vec{r}\,'(t)\,dt$

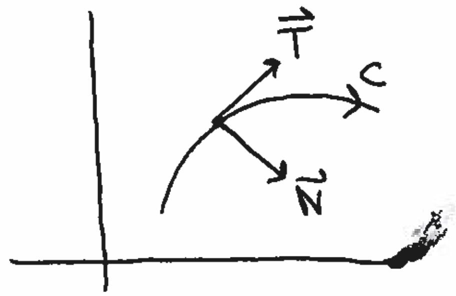
So $\quad d\vec{r} = \vec{T}\,ds = \vec{r}\,'(t)\,dt$

Work = (Force)(distance)

so

$$W = \int_C \vec{F}(x,y) \circ d\vec{r}$$

Also



$\vec{T} = \langle dx, dy \rangle$

$\vec{N} = \langle dy, -dx \rangle$

If $\vec{F} = \langle P, Q \rangle$, then

$$W = \int_C (P\,dx + Q\,dy) = \int_C \vec{F} \cdot d\vec{r}$$

$$\text{flux} = \int_C (P\,dy - Q\,dx) = \int_C \vec{F} \cdot \vec{N}\,ds$$

Flux measures the "flow" across a boundary (the curve) and is used in hydrodynamics and in biology (migration).

Example

(iii) Let $C: \vec{r}(t) = \langle t, t^2 \rangle, 0 \le t \le 1$ and $\vec{F} = \langle -y, x \rangle$, then

$$W = \int_C (-y\,dx + x\,dy)$$

$x = t,\ dx = dt$

$y = t^2,\ dy = 2t\,dt$

$$= \int_0^1 (-t^2 dt + (t)(2t)dt)$$

$$= -\tfrac{1}{3}t^3 + \tfrac{2}{3}t^3 \Big|_0^1$$

$$= \frac{1}{3}$$

and

$$\text{flux} = \int_C (-y\,dy - x\,dx)$$

$$= \int_0^1 (-t^2 \cdot 2t\,dt - t \cdot dt)$$

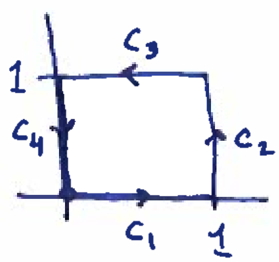
$$= -\tfrac{1}{2}t^4 - \tfrac{1}{2}t^2 \Big|_0^1$$

$$= -1$$

<u>Example</u>

(iv) Let $\vec{F} = \langle 2xy, x^2 \rangle = \nabla(x^2 y)$ and

C:



the boundary of the unit square oriented in a counter-clockwise direction.

Calculate the work.

$$W = \int_C \vec{F} \cdot d\vec{r}$$

we need to break up the integral into 4 pieces and add them up.

C_1: $\vec{r}(t) = \langle t, 0 \rangle$, $0 \le t \le 1$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 \langle 0, t^2 \rangle \cdot \langle 1, 0 \rangle\,dt = 0$$

$C_2 : \vec{r}(t) = \langle 1, t \rangle, \ 0 \le t \le 1$

$$\int_{C_2} \vec{F} \circ d\vec{r} = \int_0^1 \langle 2(1)(t), (1)^2 \rangle \circ \langle 0, 1 \rangle \, dt$$

$$= \int_0^1 1 \, dt = 1$$

$C_3 : \vec{r}(t) = \langle 1-t, 1 \rangle, \ 0 \le t \le 1$

$$\int_{C_3} \vec{F} \circ dt = \int_0^1 \langle 2(1-t)(1), (1-t)^2 \rangle \circ \langle -1, 0 \rangle \, dt$$

$$= \int_0^1 2(t-1) \, dt$$

$$= t^2 - 2t \Big|_0^1 = -1$$

$C_4 : \vec{r}(t) = \langle 0, 1-t \rangle, \ 0 \le t \le 1$

$$\int_{C_4} \vec{F} \circ d\vec{r} = \int_0^1 \langle 0, 0 \rangle \circ \langle 0, -1 \rangle \, dt = 0$$

So

$$\int_C \vec{F} \circ d\vec{r} = \int_{C_1} \vec{F} \circ d\vec{r} + \int_{C_2} \vec{F} \circ d\vec{r} + \int_{C_3} \vec{F} \circ d\vec{r} + \int_{C_4} \vec{F} \circ d\vec{r}$$

$$= 0 + 1 + (-1) + (0)$$

$$= 0$$