§16.5 Curl and Divergence

Recall: $\nabla f(x,y,z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

Operator notation: $\nabla f = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle f$

that is $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ is the gradient operator.

So ∇f is a scalar (f) "times" a vector (∇).

Look at the two other products: dot and cross product.

<u>Divergence</u> $\nabla \circ \vec{F}$

Let $\vec{F} = \langle P, Q, R \rangle$ be a vector field, then

$$\nabla \circ \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Example

(i) $\vec{F} = \langle xy^2, yz^2, \sin(xyz) \rangle$, then

$$\nabla \circ \vec{F} = \frac{\partial}{\partial x}[xy^2] + \frac{\partial}{\partial y}[yz^2] + \frac{\partial}{\partial z}[\sin(xyz)]$$

$$= y^2 + z^2 + xy\cos(xyz)$$

Divergence is a measure of "expansion" at a point. A fluid is called incompressible if the vector field modleing the fluid has the property $\nabla \cdot \vec{F} = 0$.

Curl $\nabla \times \vec{F}$

If $\vec{F} = \langle P, Q, R \rangle$ then

$$\nabla \times \vec{F} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{pmatrix}$$

$$= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

Example

(ii) $\vec{F} = \langle xy^2, yz^2, \sin(xyz) \rangle$, then

$$\nabla \times \vec{F} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & yz^2 & \sin(xyz) \end{pmatrix}$$

$$= \langle xz\cos(xyz) - 2yz^2, 0 - yz\cos(xyz), 0 - 2xy \rangle$$

Note that if $f \in C^2(D)$, that is has continuous second order derivatives in the region D, then

$$\nabla \times (\nabla f) = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{pmatrix}$$

$$= \langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy} \rangle$$

$$= \vec{0} \qquad \text{b/c } f_{yz} = f_{zy},\ f_{xz} = f_{zx},\ f_{yx} = f_{xy}$$

Recall that if $\vec{F} = \nabla f$ then $\vec{F}$ is conservative.

<u>Theorem</u> Let $\vec{F} = \langle P, Q, R \rangle$ be a vector field in $\mathbb{R}^3$ with $P, Q, R \in C^1(D)$ and $\nabla \times \vec{F} = \vec{0}$ on D, then $\vec{F}$ is conservative on D and there exists f, such that $\vec{F} = \nabla f$

<u>Example</u>

(iii) Show that $\vec{F} = \langle y - 4xz, x + 2yz^2, 2y^2z - 2x^2 \rangle$ is conservative and find a potential function f, such that $\vec{F} = \nabla f$.

$$\nabla \times \vec{F} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-4xz & x+2yz^2 & 2y^2z-2x^2 \end{pmatrix}$$

$$= \langle 4yz - 4yz, -4x - (-4x), 1-1 \rangle = \vec{0}$$

So $\vec{F}$ is conservative.

Now $\vec{F} = \nabla f \Longrightarrow$

$$f_x \overset{①}{=} y - 4xz, \quad f_y \overset{②}{=} x + 2yz^2, \quad f_z \overset{③}{=} 2y^2z - 2x^2$$

from ① $\Rightarrow$ $f(x,y,z) = xy - 2x^2z + \underbrace{K(y,z)}_{\text{"constant" of integration}}$

then

$$f_y = x + \frac{\partial K}{\partial y} \overset{②}{\Longrightarrow} \frac{\partial K}{\partial y} = 2yz^2 \overset{\text{integrate}}{\Longrightarrow} K(y,z) = y^2z^2 + J(z)$$

So $f(x,y,z) = xy - 2x^2z + y^2z^2 + J(z)$

then

$$f_z = -2x^2 + 2y^2z + J'(z) \overset{③}{\Longrightarrow} J'(z) = 0 \overset{\text{integrate}}{\Longrightarrow}$$

$J(z) = C$ a constant.

thus the family of potential functions are

$$f(x,y,z) = xy - 2x^2z + y^2z^2 + C$$

The vector forms of Green's theorem:

$\vec{F} = \langle P, Q \rangle$ note P & Q are functions of x & y

So $\nabla \circ \vec{F} = P_x + Q_y$ and

$$\nabla \times \vec{F} = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{pmatrix} = \left\langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

Recall: G.T.: $\oint_c P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

So $W = \oint_c \vec{F} \circ d\vec{r} = \iint_D (\nabla \times \vec{F}) \circ \vec{k}\, dA$

Also $\text{Flux} = \oint_c \vec{F} \circ \vec{N}\, ds = \iint_D \nabla \circ \vec{F}\, dA$

BTW curl measures the "spin" at a point.

So work around a closed curve is the sum of the "spins" inclosed in the curve and flux accross a closed boundary is the sum of the "expansion/contractions" in side the boundary.