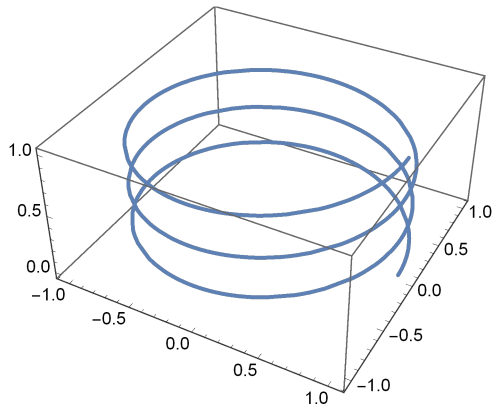
16.6 Parametric Surfaces and their Areas

These notes are intended as a supplement to the material covered in section 16.6 of the book. Please read the material in the book before proceeding.

Recall parametric curves, $C: \vec{r}(t) = \langle x(t), y(t), z(t)\rangle$. This is a vector function with one independent variable and produces a curve. For example:

In[1]:=
```
ParametricPlot3D[{Cos[t], Sin[t], t / (6 Pi)}, {t, 0, 6 Pi}]
```

Out[1]=



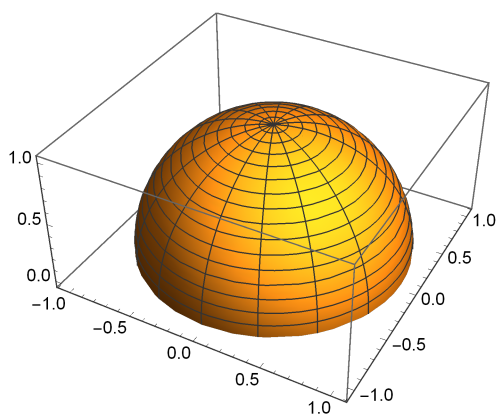
This is a helix.

A little piece of arc-length is given by $\Delta s = \left|\vec{r}\,'(t)\right| \Delta t$.

Now look at a vector function of two independent variables, $S: \vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$. This is a surface. For example:

In[2]:=
```
ParametricPlot3D[{Sin[u] Cos[v], Sin[u] Sin[v], Cos[u]}, {u, 0, Pi / 2}, {v, 0, 2 Pi}]
```

Out[2]=



This is the hemisphere or upper half of a sphere of radius 1. Recall that the change of variables between rectangular and spherical was $x = \rho \operatorname{Sin}[\phi] \operatorname{Cos}[\theta]$, $y = \rho \operatorname{Sin}[\phi] \operatorname{Sin}[\theta]$, and $z = \rho \operatorname{Cos}[\phi]$, where ρ is distance from the origin, ϕ is measured down from the positive z-axis and θ is the same angle as in polar (or cylindrical) coordinates.

Recall from section section 15.6 that when a surface is given by $z = f(x, y)$, then the surface area was

$$\Delta S = \left|\vec{a} \times \vec{b}\right| = \sqrt{f_x(x, y)^2 + f_y(x, y)^2 + 1}\ \Delta x \Delta y$$

where $\vec{a} = \langle \Delta x, 0, f_x(x, y)\rangle$ and $\vec{b} = \langle 0, \Delta y, f_y(x, y)\rangle$. In a similar fashion, the area of a parametric surface is given by

$$\Delta S = \left|\vec{r_u} \times \vec{r_v}\right| \Delta u \Delta v.$$

Then the total surface area would be the double integral

$$\iint_D \left|\vec{r_u}(u, v) \times \vec{r_v}(u, v)\right| du\, dv,$$

where D is the (u, v) region that maps the part of the surface of interest.

Example (i)

Find a parameterization for the surface $2x + y + z = 6$ and use the parameterization to find the area of the part of the surface that is in the first octant.

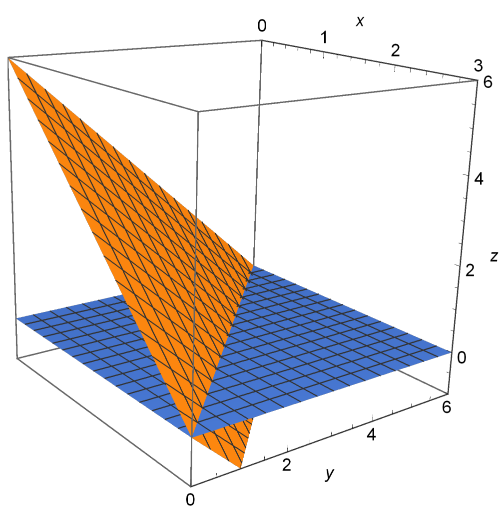
First: because we can write the equation as z as a function of x and y, we can use x and y as the parameters. Thus, one parameterization of the surface is

$$\vec{r}(x, y) = \langle x, y, 6 - 2x - y\rangle.$$

Next, look at a graph

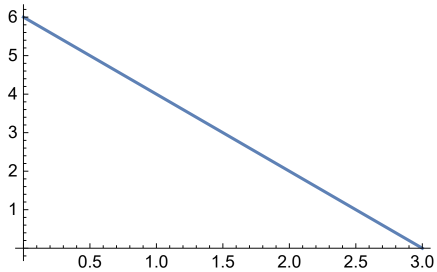
In[3]:=
```
ParametricPlot3D[{{x, y, 6 - 2 x - y}, {x, y, 0}}, {x, 0, 3}, {y, 0, 6},
 PlotRange → {{0, 3}, {0, 6}, {-1, 6}}, AxesLabel → {x, y, z}, BoxRatios → {1, 1, 1}]
```

Out[3]=



The part of the surface that is in the first octant has parameter values in a triangular region:

In[4]:= `Plot[6 - 2 x, {x, 0, 3}]`

Out[4]=


The “increment” of surface area is

$$d S = \left|\vec{r}_x \times \vec{r}_y\right| \, d x \, d y = |\langle 1, 0, -2\rangle \times \langle 0, 1, -1\rangle| \, d x \, d y = |\langle 2, 1, 1\rangle| \, d x \, d y.$$

Note that in this case, $\vec{r}_x \times \vec{r}_y = \langle 2, 1, 1\rangle$, the normal vector given by the coefficients of the equation of the plane in standard form. So the area of the surface is the double integral:

$$\int_0^3 \int_0^{6-2x} |\langle 2, 1, 1\rangle| \, dy \, dx = \sqrt{2^2 + 1^2 + 1^2} \left(\frac{1}{2}(3)(6)\right) = 9\sqrt{6}.$$

Note that we do not need to actually do the integral (of a constant), because the value of the integral is the area of the region times the (constant) integrand.

Example (ii)

Find a parameterization of the top half of $x^2 + y^2 - z^2 = 1$.

Note that we can rewrite the equation as $x^2 + y^2 = z^2 + 1$. So, this is a hyperboloid in one sheet. For fixed z, we have a circle of radius $\sqrt{z^2 + 1}$. Thus, one parameterization is

$$\vec{r}(\theta, z) = \left\langle \sqrt{z^2+1} \, \mathrm{Cos}[\theta], \sqrt{z^2+1} \, \mathrm{Sin}[\theta], z \right\rangle, \; 0 \le \theta \le 2\pi, \; z \ge 0.$$

The increment of surface area for this one is a bit more complicated:

$$\vec{r}_\theta = \left\langle -\sqrt{z^2+1} \, \mathrm{Sin}[\theta], \sqrt{z^2+1} \, \mathrm{Cos}[\theta], 0 \right\rangle,$$

$$\vec{r}_z = \left\langle \frac{z}{\sqrt{z^2+1}} \mathrm{Cos}[\theta], \frac{z}{\sqrt{z^2+1}} \mathrm{Sin}[\theta], 1 \right\rangle,$$

so we have

$$\left|\vec{r}_\theta \times \vec{r}_z\right| = \left|\left\langle \sqrt{z^2+1} \, \mathrm{Cos}[\theta], \sqrt{z^2+1} \, \mathrm{Sin}[\theta], -z \right\rangle\right| = \sqrt{2 z^2 + 1}.$$