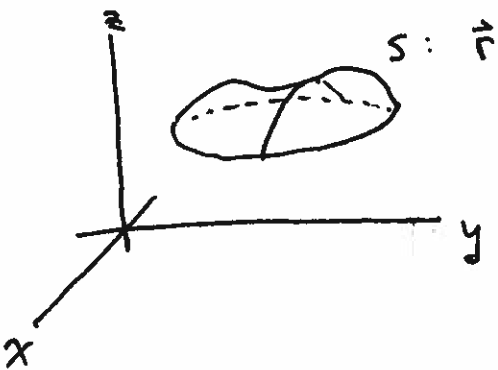
§16.7 Surface Integrals

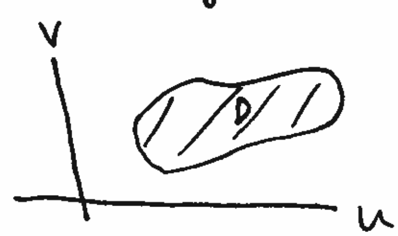
$$\iint_S f(x,y,z)\,dS = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{i,j}^*)\,\Delta S_{i,j}$$

$$= \iint_D f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,du\,dv$$

where S is a surface paraterized by $\vec{r}(u,v)$, $(u,v) \in D$:

$S: \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$





Also, $P_{i,j}^*$ are points on the surface S and $\Delta S_{i,j}$ is an increment of surface area.

This is basically the same as line integrals.

Example

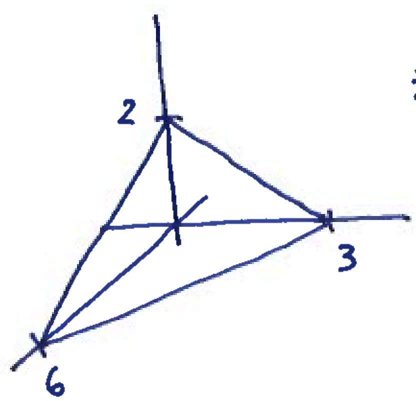
(i) $f(x, y, z) = 2x + yz$

S : the first octant part of

$$x + 2y + 3z = 6$$

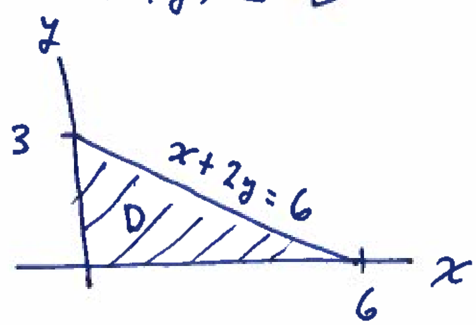
Find $I = \iint_S f(x, y, z)\, dS$

the region :



$$z = \frac{6 - x - 2y}{3} \iff \vec{r}(x, y) = \langle x, y, 2 - \tfrac{1}{3}x - \tfrac{2}{3}y \rangle$$

where $(x, y) \in D$



$$dS = |\vec{r}_x \times \vec{r}_y|\, dx\,dy = |\langle 1, 0, -\tfrac{1}{3} \rangle \times \langle 0, 1, -\tfrac{2}{3} \rangle|\, dx\,dy$$

$$= \frac{\sqrt{14}}{3}\, dx\,dy$$

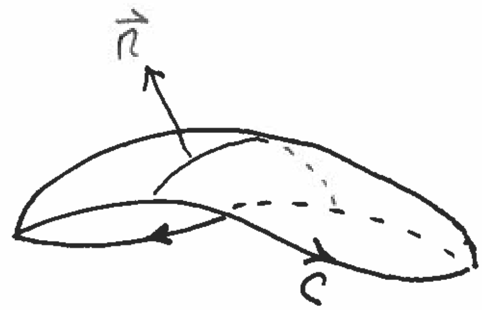
then

$$I = \int_0^3 \int_0^{6-2y} \left(2x + y\left(2 - \tfrac{1}{3}x - \tfrac{2}{3}y\right)\right)\left(\frac{\sqrt{14}}{3}\right) dx\,dy$$

$\vdots$

$$= 9\left(\sqrt{\tfrac{7}{2}} + \sqrt{14}\right) \approx 50.512$$

Oriented Surfaces



Oriented upward
(upward normal)
positive orientation
w.r.t. the boundary
curve C.

Mobius strip is
not orientable
(only one "side" of surface)

$$\vec{N} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \qquad dS = |\vec{r}_u \times \vec{r}_v| \, du\,dv$$

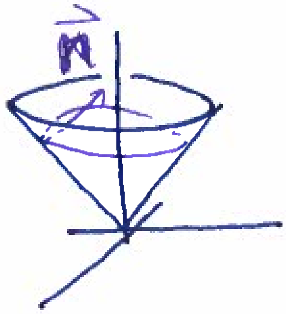
so $\vec{N}\, dS = d\vec{S} = (\vec{r}_u \times \vec{r}_v)\, du\,dv$

(choose the cross product that produces the upward or outward normal vector.)

Example

(ii) Find the flux across the suface S: $z=\sqrt{x^2+y^2}$, $0 \le z \le 1$ where $\vec{F} = \langle x, y, z^4 \rangle$

$z=\sqrt{x^2+y^2}$ is a cone. In cylindrical this is $z=r$.



so S: $\vec{r}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), r \rangle$

$$\vec{r}_r = \langle \cos(\theta), \sin(\theta), 1 \rangle$$

$$\vec{r}_\theta = \langle -r\sin(\theta), r\cos(\theta), 0 \rangle$$

$$\vec{r}_r \times \vec{r}_\theta = \langle -r\cos(\theta), -r\sin(\theta), \underset{\uparrow\, r>0 \Rightarrow \text{up}}{r} \rangle$$

the region of integration is $0 \le \theta \le 2\pi$, $0 \le r \le 1$.

So

$$\text{flux} = \iint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^1 \langle r\cos(\theta), r\sin(\theta), r^4 \rangle \cdot \langle -r\cos(\theta), -r\sin(\theta), r \rangle \, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(-r^2\cos^2(\theta) - r^2\sin^2(\theta) + r^5\right) dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 (r^5 - r^2)\, dr\, d\theta$$

$$= 2\pi\left(\frac{1}{6} - \frac{1}{3}\right) = -\frac{\pi}{3}$$

So the flow is generaly down and out across the surface.