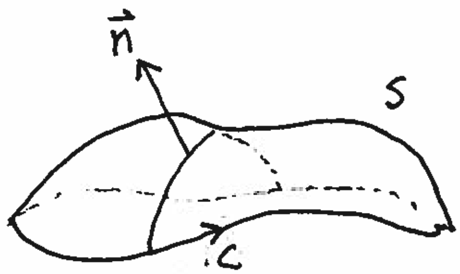
§16.8 Stoke's Theorem

Read the statement of the theorem on page 1146 of the book.



Let S be an oriented surface bounded by a P.O.P.S.S.C.C., C. then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot d\vec{S}$$

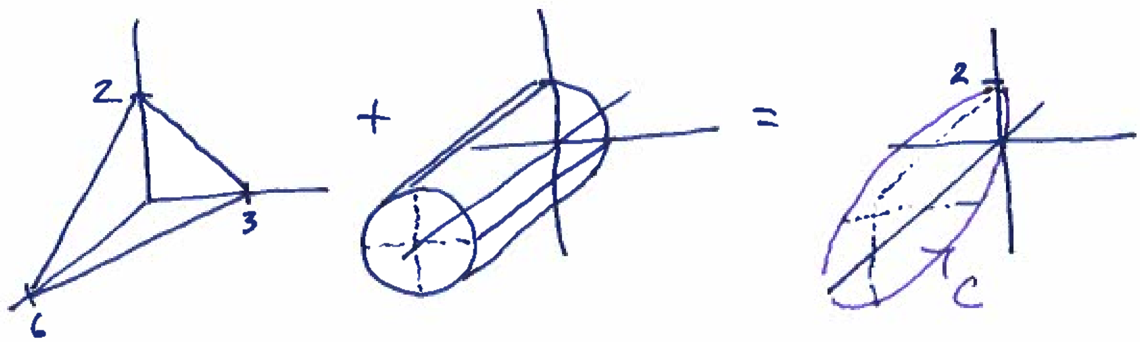
other notation

$$\oint_C \vec{F} \cdot \vec{T} \, ds = \iint_S \nabla \times \vec{F} \cdot \vec{N} \, dS$$

Stoke's theorem is an extension of Green's theorem. That is, Green's theorem is a special case of Stoke's theorem.

Example

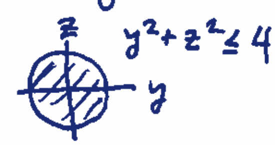
(i) Find $\oint_C \langle -z, y, x\rangle \cdot d\vec{r}$ where C is the positively oriented curve of intersection between the plane $x+2y+3z=6$ and the cylinder $y^2+z^2=4$.



The curve C bounds the part of the plane $x=6-2y-3z$ that is inside $y^2+z^2=4$

If we let $f(x,y,z) = x+2y+3z-6$, then

$$\vec{N}\,dS = \nabla f \, dy\,dz = \langle 1,2,3\rangle \, dy\,dz$$

$\nabla\times\langle -z, y, x\rangle = \langle 0, -2, 0\rangle$ and D: 

so

$$\oint_C \langle -z,y,x\rangle\, d\vec{r} \overset{S.T.}{=} \iint_S (\nabla\times\langle -z,y,x\rangle)\cdot \vec{N}\,dS$$

$$= \iint_D (-4)\, dy\,dz$$

$$= (-4)(4\pi) = -16\pi$$

Example

(ii) Let $F = \langle xz, yz, x^2+y^2+z^2 \rangle$

$S: z = \sqrt{16 - x^2 - y^2}$ the top half of a sphere of radius 4 centered at the origin, and C be the boundry of S. Find $\oint_C \vec{F} \cdot d\vec{r}$.

The curve has a parameterization of

$C: \vec{r}(t) = \langle 4\cos(t), 4\sin(t), 0 \rangle, \ 0 \le t \le 2\pi$

So as a line integral we have

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle 0, 0, 4 \rangle \cdot \langle -4\sin(t), 4\cos(t), 0 \rangle dt$$

$$= \int_0^{2\pi} 0 \, dt = 0$$

If we apply Stoke's theorem, then don't use the S that is given. Instead use the disk $x^2 + y^2 \le 16$ and $z = 0$. Then

$\nabla \times \vec{F} = \langle y, -x, 0 \rangle$ and $\vec{N} dS = \vec{k} \, dx dy$. So

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \vec{N} dS = \iint_{x^2+y^2 \le 16} \langle y, -x, 0 \rangle \cdot \langle 0, 0, 1 \rangle dx dy$$

$$= \iint_{x^2+y^2 \le 16} 0 \, dx dy = 0$$

Note that in this example either the line integral or the surface integral (for the surface we chose) was relatively easy. We don't need to use the surface given, especially if there is an "easier" surface available that is also bounded by the curve.