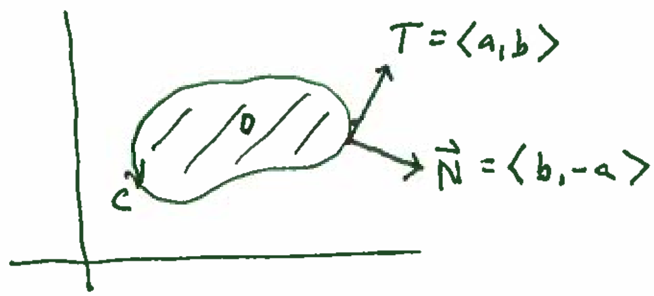
§16.9 The Divergence Theorem

Recall Green's Theorem:



$\vec{F} = \langle P, Q \rangle$

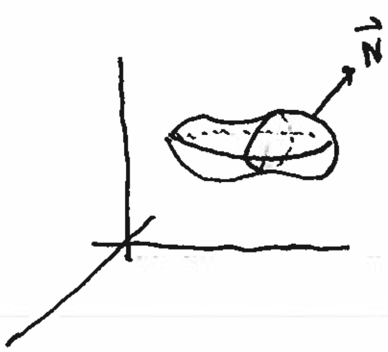
$$\oint_C \vec{F} \circ \vec{N}\, ds = \oint_C \langle P, Q \rangle \circ \langle dy, -dx \rangle$$

$$= \oint_C P\,dy - Q\,dx$$

$$\overset{G.T.}{=} \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA$$

$$= \iint_D \nabla \circ F\, dA$$

The Divergence Theorem (page 1153 of book) is an extension to $\mathbb{R}^3$ of Green's Theorem.

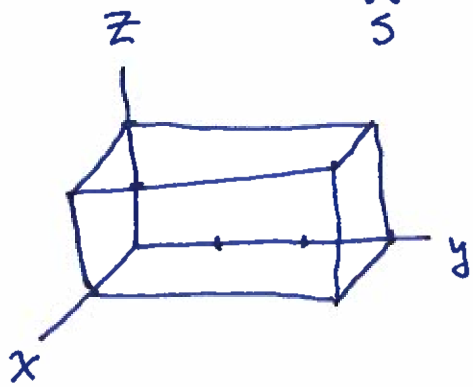


$$\iint_S \vec{F} \circ \vec{N}\, dS = \iiint_D \nabla \circ \vec{F}\, dV$$

Example

(i) Let $\vec{F} = \langle x, xy, xyz \rangle$ and D be the region $0 \le x \le 1$, $0 \le y \le 3$, $0 \le z \le 2$ with S being the oriented surface that bounds D. Find flux:

$$\iint_S \vec{F} \cdot \vec{N} \, dS$$



As a surface integral we would need to split the surface into six parts. Applying the Div. Thrm. we get:

$$\iint_S \vec{F} \cdot \vec{N} \, dS = \int_0^2 \int_0^3 \int_0^1 (1 + x + xy) \, dx \, dy \, dz$$

$$= \int_0^2 \int_0^3 \int_0^1 1 \, dx \, dy \, dz + \int_0^2 \int_0^3 \int_0^1 x \, dx \, dy \, dz + \int_0^2 \int_0^3 \int_0^1 xy \, dx \, dy \, dz$$

$$= (2-0)(3-0)(1-0) + (2-0)(3-0) \int_0^1 x \, dx + (2-0)\left(\int_0^3 y \, dy\right)\left(\int_0^1 x \, dx\right)$$

by using properties from section 15.2

$$= 6 + 6\left(\tfrac{1}{2}x^2\Big|_0^1\right) + 2\left(\tfrac{1}{2}y^2\Big|_0^3\right)\left(\tfrac{1}{2}x^2\Big|_0^1\right)$$

$$= 6 + 3 + \frac{9}{2}$$

$$= 13.5$$

Example

(ii) $\vec{F} = \langle xy^2, yz^2, x^2z \rangle$, S is the sphere of radius 1: $x^2+y^2+z^2=1$ with outward (positively oriented) $\vec{N}$. Find the flux out.

$$\iint_S \vec{F}\cdot\vec{N}\,dS = \iiint_{x^2+y^2+z^2\le 1} (y^2+z^2+x^2)\,dV$$

Use spherical coordinates

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2(\rho^2\sin(\varphi))\,d\rho\,d\phi\,d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi}\sin(\varphi)\,d\varphi\right)\left(\int_0^1 \rho^4\,d\rho\right)$$

$$= (2\pi)\left(-\cos(\varphi)\Big|_0^{\pi}\right)\left(\tfrac{1}{5}\rho^5\Big|_0^1\right)$$

$$= (2\pi)\left(-(-1)+(1)\right)\left(\tfrac{1}{5}-0\right)$$

$$= \frac{4\pi}{5}$$