Chapter 1

Theorem 2 there is no rational number whose square is 2.

Proof: Prime factorizations are unique.

Assume for the sake of contradiction that there is a rational number whose square is 2.

That is, $\exists\, r, s \in \mathbb{N} \ni \left(\frac{r}{s}\right)^2 = 2$.

r and s have prime factorizations. Let

$$r = (2^{n_1})(3^{n_2})(5^{n_3})(7^{n_4})\cdots \quad \text{and}$$

$$s = (2^{m_1})(3^{m_2})(5^{n_3})(7^{m_4})\cdots$$

where $n_k, m_j \in \mathbb{N} \cup \{0\}$ for all k, j.

Then $\left(\frac{r}{s}\right)^2 = 2 \Rightarrow r^2 = 2s^2 \Rightarrow$

$$(2^{2n_1})(3^{2n_2})(5^{n_3})\cdots = (2^{2m_1+1})(3^{2m_2})(5^{2m_3})\cdots$$

$\Rightarrow 2n_1 = 2m_1 + 1,\ 2n_2 = 2m_2,\ 2m_3 = 2n_3, \ldots$

But $2n_1$ is even and $2m_1+1$ is odd $\Rightarrow\Leftarrow$

$\therefore$ the assumption is false

$\blacksquare$

Example 6

- What is $\emptyset^c$?

  Depends on the Universal set. For us $\emptyset^c = \mathbb{R}$.

- What is $\mathbb{R}^c$?

  For us $\mathbb{R}^c = \emptyset$

- What is $\mathbb{Q}^c$?

  $\mathbb{Q}^c$ is the set of irrational numbers

- What is $\mathbb{Z}^c$

  $\mathbb{Z}^c = \{x \in \mathbb{R} \mid x \notin \mathbb{Z}\}$

  Some examples: $\sqrt{2} \in \mathbb{Z}^c$, $\frac{2}{3} \in \mathbb{Z}^c$, $\pi \in \mathbb{Z}^c$, $-\frac{7}{2} \in \mathbb{Z}^c$.

Example 8

For each $n \in \mathbb{N}$, $A_n = \{n, n+1, n+2, \ldots\}$. Verify

- $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$

  Proof (By induction)

  (BC) $A_2 \subseteq A_1$: clearly

  $\{2, 3, 4, \ldots\} \subseteq \{1, 2, 3, 4, \ldots\}$

  (IH) Assume that $\exists\, j \in \mathbb{N} \ni \cancel{A_{j+1} \subseteq A_j}$

  $A_j \subseteq A_{j-1} \subseteq A_{j-2} \subseteq \cdots \subseteq A_1$

  (IS) Show that (IH) $\Rightarrow$

  $A_{j+1} \subseteq A_j \subseteq A_{j-1} \subseteq \cdots \subseteq A_1$

Since $A_j = A_{j+1} \cup \{j\}$, then $x \in A_{j+1} \Rightarrow x \in A_j$
$\Rightarrow A_{j+1} \subseteq A_j$. Thus by the (IH)

$$A_{j+1} \subseteq A_j \subseteq A_{j-1} \subseteq \cdots \subseteq A_1$$

$\therefore$ by Strong Induction

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$$

► $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$

<u>Proof</u>

Let $x \in \mathbb{R}$. $\exists\, j \in \mathbb{N} \ni x < j$ (Why?).
By definition $x \notin A_j$. $\therefore x \notin \bigcap_{n \in \mathbb{N}} A_n$.
Since $x \in \mathbb{R}$ was choosen arbitrarily, then
$\bigcap_{n \in \mathbb{N}} A_n = \mathbb{R}^c = \emptyset$.

► $\bigcup_{n \in \mathbb{N}} A_n = A_1$

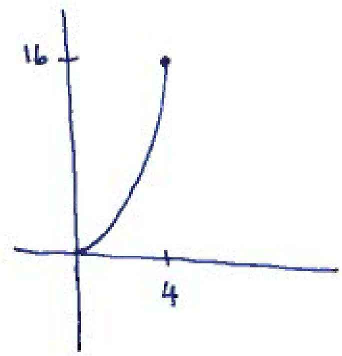
Should follow directly from the first statement and the definition of union.
That is

$$A \subseteq B \Rightarrow A \cup B = B$$

Example 15

$f : [0,4] \to \mathbb{R}$ by $f(x) = x^2$, then

- $f^{-1}([0,1]) = [0,1]$
- $f^{-1}([-1,1]) = [0,1]$
- $f^{-1}(\mathbb{R}) = [0,4]$



Note that if $g : [0, 2\pi) \to [-1,1]$ by $g(x) = \sin(x)$, then

- $g^{-1}(\frac{1}{2}) = \{ \frac{\pi}{6}, \frac{5\pi}{6} \}$
- $g^{-1}([-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]) = [0, \frac{\pi}{4}] \cup [\frac{3\pi}{4}, \frac{5\pi}{4}] \cup [\frac{7\pi}{4}, 2\pi)$