Theorem 46

1) $\mathbb{Q}$ is countable
2) $\mathbb{R}$ is uncountable

Proof

1) WOLG look at $\mathbb{Q}^+$. List the ellements in $\mathbb{Q}^+$ in a table:

$$\begin{array}{cccccc} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ 2 & \frac{2}{3} & \frac{2}{5} & \frac{2}{7} & \frac{2}{9} & \cdots \\ 3 & \frac{3}{2} & \frac{3}{4} & \frac{3}{5} & \frac{3}{7} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}$$

Define $f : \mathbb{N} \to \mathbb{Q}^+$ by $f(1) = 1$, $f(2) = \frac{1}{2}$, $f(3) = 2$, $f(4) = \frac{1}{3}$, $f(5) = \frac{2}{3}$, $f(6) = 3$, $f(7) = \frac{1}{4}$, $f(8) = \frac{2}{5}$, $f(9) = \frac{3}{2}$, ...

f is 1-1 and onto, $\therefore$ $\mathbb{N} \sim \mathbb{Q}^+$. ▨

2) WOLG look at $S = \{x \in \mathbb{R} \mid 0 < x < 1\}$ and assume S is countable. Write each number in binary: for $x_k \in S$,

$$x_k = 0.d_{1k}d_{2k}d_{3k}\dots$$

where $d_{jk} \in \{0,1\}$. Now consider

$$x = 0.c_1c_2c_3\dots \in S$$

where $$c_j = \begin{cases} 0 & \text{if } d_{jj} = 1 \\ 1 & \text{if } d_{jj} = 0 \end{cases}$$

Clearly $x \neq x_j$ for $j = 1, 2, 3, \ldots$ since they differ in the j^{th} digit. $\therefore x \notin S \Rightarrow\Leftarrow$

Thus S is uncountable. Not $S \subseteq \mathbb{R} \Rightarrow$ $\mathbb{R}$ is not countable.

$\blacksquare$