Chapter 2: Approximating Solutions of Linear Systems

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Overview

Linear Operators

An **operator** is a relation between one set (of functions usually) to another set.

Example 1

Differentiation and indefinite integrals are examples of operators between functions. Matrix multiplication is an operator between vector spaces.

Definition 2

Let $L : A \to B$ be an operator with $f, g \in A$ and c be in the set of scalars. L is said to be **linear** if

1. $L[c\, f] = c\, L[f]$ and
2. $L[f + g] = L[f] + L[g]$.

Linear Systems of Equations

$$
\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= b_2 \\
&\vdots \\
a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &= b_m
\end{aligned}
$$

can be written in matrix form as

$$
\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \text{ or } A\vec{x} = \vec{b}
$$

where $a_{i,j}$ and b_i are given constants and x_i are unknowns we are trying to find.

Examples of Systems of Equations

Example 3

$$\begin{aligned} 2x - y &= 4 \\ x + y &= 5 \end{aligned}$$

is an example of a linear system of equations.

Example 4

$$\begin{aligned} x^2 + y^2 &= 4 \\ xy &= 1 \end{aligned}$$

is an example of a non-linear system of equations.

Augmented Matrices

Definition 5

Let $A\vec{x} = \vec{b}$ be a linear system of equations. The **augmented coefficient matrix** for this system is

$$\bar{A} = [A|\vec{b}]$$

Example 6

The augmented coefficient matrix for the system

$$\begin{aligned} 2x - y &= 4 \\ x + y &= 5 \end{aligned}$$

is

$$\begin{bmatrix} 2 & -1 & \vdots & 4 \\ 1 & 1 & \vdots & 5 \end{bmatrix}$$

Row Operations

Theorem 7

The solution set of systems of linear equations are invariant under the following operations on their augmented matrices:

1. *swapping two different rows,*
2. *multiplication of terms in a row by a non-zero constant,*
3. *addition of one row to another (different) row.*

By combining the last two, we can add a non-zero multiple of one row in an augmented matrix to another row without changing the solution set of the corresponding system of linear equations.

Types of Solutions to Linear Systems of Equations

Definition 8

A linear system of equations is called **consistent** if it has a unique solution, **inconsistent** it it has no solutions and **dependent** if it has an infinite number of solutions.

Each system of linear equations has one unique solution status defined above.

Overview

Basic Elimination

Example 9

Solve the following system of linear equations.

$$\begin{aligned} x + 2y + z &= 3 \\ 2x - y + z &= 2 \\ -x - 2y + 3z &= -7 \end{aligned}$$

Develop an algorithm (in Matlab) which has as impute the augmented coefficient matrix for a system of linear equations and outputs the reduced row-echelon form for the system.

Operation Count

Question: How many multiplications (divisions) are required to reduce the augmented matrix for an $n \times n$ system of linear equations to an upper triangular form?

Question: how many multiplications (divisions) are required to take the augmented matrix (in upper triangular form) and perform the back-substitution?

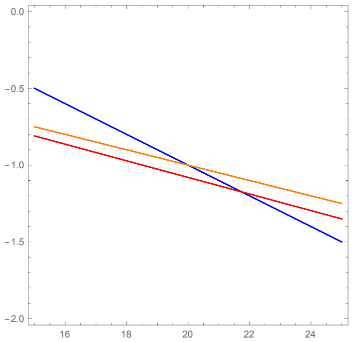
Homework

Homework assignment 3, due: TBA

How Error in Representation Effects Solutions

Example 10

The equation $x + 10y = 10$ is graphed in blue, $0.054x + y = 0$ in red, and $0.05x + y = 0$ in orange. Note how a 0.004 diference in representation produces a large change in solution.



Basic Pivoting

Example 11

Reduce the following augmented coefficient matrix.

$$\begin{bmatrix} 0 & 0 & -3 & 1 & \vdots & 2 \\ 2 & 1 & -5 & 1 & \vdots & -4 \\ 2 & 1 & 10 & -2 & \vdots & 8 \\ 3 & 35 & 20 & 1 & \vdots & 4 \end{bmatrix}$$

In this case we choose the first non-zero element in the pivot column as the pivot element.

Partial Pivoting

Example 12

Reduce the following augmented coefficient matrix using partial pivoting.

$$\left[\begin{array}{cccc|c} 0 & 0 & -3 & 1 & 2 \\ 2 & 1 & -5 & 1 & -4 \\ 2 & 1 & 10 & -2 & 8 \\ 3 & 35 & 20 & 1 & 4 \end{array}\right]$$

In this case we choose the largest (in magnitude) element in the pivot column as the pivot element.

Scaled Partial Pivoting

Let s_i be the maximum of $|a_{i,j}|$ for $1 \leq j \leq n$. Then the pivot element for the pivot column is in row p, where p is the least value such that

$$\frac{|a_{p,i}|}{s_p} = \max_{i \leq k \leq n} \left\{ \frac{|a_{k,i}|}{s_k} \right\}$$

Example 13

Reduce using scaled partial pivoting

$$\begin{bmatrix} 0 & 0 & -3 & 1 & \vdots & 2 \\ 2 & 1 & -5 & 1 & \vdots & -4 \\ 2 & 1 & 10 & -2 & \vdots & 8 \\ 3 & 35 & 20 & 1 & \vdots & 4 \end{bmatrix}$$

Homework

Homework assignment 4, due: TBA

Overview

Matrix Inversion

Overview

Determinants

Overview

Norms of Vectors and Matrices

Overview

Eigenvalues and Eigenvectors