Chapter 4: Functional Limits and Continuity

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Overview

Dirichlet's Function

Example 1

Let $f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$

This function is nowhere continuous on $\mathbb{R}$. (Trust me.)

Modified Dirichlet's Function

Example 2

Let $f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$

This function is continuous only at $c = 0$. (Trust me.)

Thomae's Function

Example 3

Let $f(x) =$
$$\begin{cases} 1, & \text{if } x = 0 \\ \frac{1}{n}, & \text{if } x = \frac{m}{n} \in \mathbb{Q} \backslash \{0\} \text{ is in lowest terms with } n > 0 \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

This function is continuous on the irrationals. It is discontinuous at the rationals. (Trust me.)

Overview

$\epsilon - \delta$ Version of Limit of a Function

Remark 4

Recall from Chapter 3:

- *Definition: A point x is a* ***limit point*** *of a set A if every $\epsilon-$neighborhood $V_\epsilon(x)$ of x intersects the set A in some point other than x.*
- *Theorem: A point x is a limit point of a set A if and only if $x = \lim a_n$ for some sequence (a_n) contained in A satisfying $a_n \neq x$ for all $n \in \mathbb{N}$.*
- *A point $x \in A$ is* ***isolated*** *if it is not a limit point of A.*

Definition 5

Let $f : A \mapsto \mathbb{R}$, and let c be a limit point of the domain A. We say that $\lim_{x \to c} f(x) = L$ provided that, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - L| < \epsilon$.

Topological Version of Limit of a Function

Remark 6

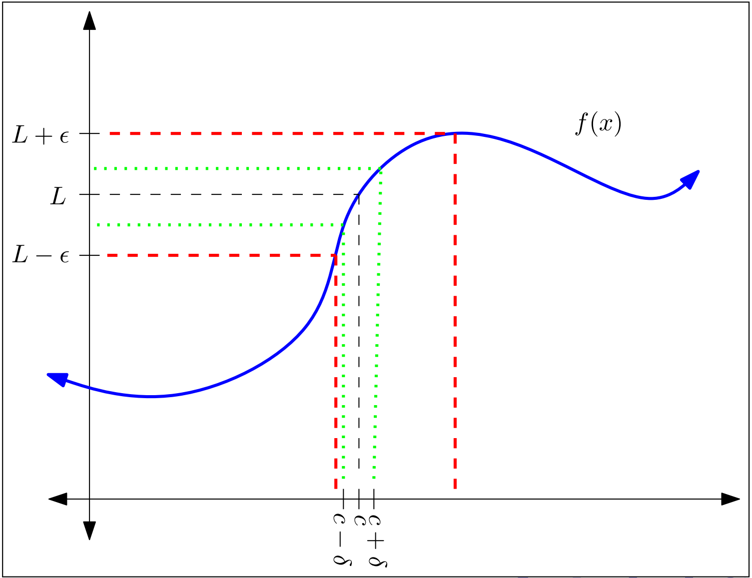
Recall:

- *Let* $a > 0$. *Then* $V_a(x) = \{y \in \mathbb{R} \mid |x - y| < a\}$.
- *The statement* $|x - c| < \delta$ *is equivalent to* $x \in V_\delta(c)$.
- *The statement* $|f(x) - L| < \epsilon$ *is equivalent to* $f(x) \in V_\epsilon(L)$.

Definition 7

Let c be a limit point of the domain of $f : A \mapsto \mathbb{R}$. We say $\lim_{x \to c} f(x) = L$ provided that, for every $\epsilon-$neighborhood $V_\epsilon(L)$ of L, there exists a $\delta-$neighborhood $V_\delta(c)$ around c whith the property that for all $x \in V_\delta(c)$ around c with the property that for all $x \in V_\delta(c)$ different from c (with $x \in A$) it follows that $f(x) \in V_\epsilon(L)$.

The Picture



Example

Example 8

Let $f(x) = 5x - 3$. Prove $\lim_{x \to 1} f(x) = 2$.

Example 9

Let $g(x) = x^2 + 1$. Prove $\lim_{x \to -2} g(x) = 5$.

Sequential Criterion for Functional Limits

Theorem 10 (Sequential Criterion for Functional Limits)

Given a function $f : A \mapsto \mathbb{R}$ and a limit point c of A, the following are equivalent

- $\lim_{x \to c} f(x) = L$
- *For all sequences* $(x_n) \subseteq A$ *satisfying* $x_n \neq c$ *and* $(x_n) \xrightarrow[n\to\infty]{} c$, *it follows that* $f(x_n) \xrightarrow[n\to\infty]{} L$.

Proof:

Algebraic Limit Theorem for Functional Limits

Corollary 11

Let f and g be functions defined on a domain $A \subseteq \mathbb{R}$, and assume $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$ for some limit point c of A. Then,

1. $\lim_{x \to c} kf(x) = kL$ *for all* $k \in \mathbb{R}$.
2. $\lim_{x \to c} [f(x) + g(x)] = L + M$.
3. $\lim_{x \to c} [f(x)g(x)] = LM$.
4. $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ *provided* $M \neq 0$.

Divergence Criterion for Functional Limits

Corollary 12 (Divergence Criterion for Functional Limits)

Let f be a function defined on A, and let c be a limit point of A. If there exist two sequences (x_n) and (y_n) in A with $x_n \neq c$ and $y_n \neq c$ for all n, and

$$\lim x_n = \lim y_n = c \text{ but } \lim f(x_n) \neq \lim f(y_n),$$

then we can conclude that the functional limit $\lim_{x \to c} f(x)$ does not exist.

Example

Example 13

Let $f(x) = \sin\left(\frac{1}{x}\right)$. Assume the usual properties of the sine function. Show that $\lim_{x\to 0} f(x)$ does not exist.

Homework

Pages: 108–109

Problems: 4.2.1, 4.2.2, 4.2.3, 4.2.5, 4.2.7, 4.2.9

Overview

Definition of Continuity

Definition 14

- A function $f : A \mapsto \mathbb{R}$ is **continuous at a point $c \in A$** if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - f(c)| < \epsilon$.
- If f is continuous at every point in A, the we say that **f is continuous on A**.

Remark 15

We can't shorten the definition of continuity to $\lim_{x \to c} f(x) = f(c)$ because the definition of functional limits requires the point c to be a limit point of A. Note that limit points of A need not be elements of A. In the definition of continuity, this is not assumed. (Isolated points of A are not limit points).

Characterizations of Continuity

Theorem 16 (Characterizations of Continuity)

Let $f : A \mapsto \mathbb{R}$, and let $c \in A$ be a limit point of A. The function f is continuous at c if and only if any one of the following conditions is met:

- *For all $\epsilon > 0$, there exists $\delta > 0$ such that $|x - c| < \delta$ (and $x \in A$) implies $|f(x) - f(c)| < \epsilon$.*
- $\lim\limits_{x \to c} f(x) = f(c)$.
- *For all $V_\epsilon(f(c))$, there exists a $V_\delta(c)$ with the property that $x \in V_\delta(c)$ (and $x \in A$) implies $f(x) \in V_\epsilon(f(c))$.*
- *If $(x_n) \xrightarrow[n\to\infty]{} c$ (with $x_n \in A$), then $f(x_n) \xrightarrow[n\to\infty]{} f(c)$.*

Proof:

Showing Discontinuity at a Point

Corollary 17 (Criterion for Discontinuity)

Let $f : A \mapsto \mathbb{R}$ and let $c \in A$ be a limit point of A. If there exists a sequence $(x_n) \subseteq A$ where $(x_n) \xrightarrow[n\to\infty]{} c$ but such that $f(x_n)$ does not converge to $f(c)$, we may conclude that f is not continuous at c.

Algebraic Continuity Theorem

Theorem 18 (Algebraic Continuity Theorem)

Assume $f : A \mapsto \mathbb{R}$ and $g : A \mapsto \mathbb{R}$ are continuous at a point $c \in A$. Then,

- *$kf(x)$ is continuous at c for all $k \in \mathbb{R}$.*
- *$f(x) + g(x)$ is continuous at c.*
- *$f(x)g(x)$ is continuous at c*
- *$\frac{f(x)}{g(x)}$ is continuous at c, provided the quotient is defined.*

Example

Example 19

Show the following functions are continuous.

- $f(x) = k$, for any $k \in \mathbb{R}$.
- $g(x) = x$.
- $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, where $a_i \in \mathbb{R}$ for $0 \leq i \leq n$.
- All rational functions (quotients of polynomials) are continuous over their domains.

Remark 20

Fact: $f(x) = \sqrt{x}$ is continuous on its domain. (See book for proof).

Example

Example 21

Show that $f(x) = x \sin\left(\frac{1}{x}\right)$ is continuous at $x = 0$.

Composition of Functions

Theorem 22 (Composition of Continuous Functions)

Given $f : A \mapsto \mathbb{R}$ and $g : B \mapsto \mathbb{R}$, assume that the range $f(A) = \{f(x) \mid x \in A\} \subseteq B$. If f is continuous at $c \in A$, and if g is continuous at $f(c) \in B$, then

$$(g \circ f)(x) = g[f(x)]$$

is continuous at c.

Proof:

Example

Example 23

Note that $f(x) = \sqrt{x^2 - 7x + 5}$ is continuous on it's domain since polynomials and the square root function are both continuous, and the composition of continuous functions remains continuous (on the restricted domain).

Homework

Pages: 113–114

Problems: 4.3.1, 4.3.4, 4.3.6, 4.3.7, 4.3.9, 4.3.10, 4.3.12

Overview

Definition

Remark 24

Recall: A sequence (x_n) is ***bounded*** *if there exists a number $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.*

Definition 25

Let $f : A \mapsto \mathbb{R}$. We say that f is **bounded on a set B**, where $B \subseteq A$ if there exists an $M > 0$ such that for every sequence $(x_n) \subseteq B$, we have $|f(x_n)| \leq M$ for all n.

Example

Remark 26

We wish to determine conditions that preserve conditions on sets under continuous functions. That is, if f is continuous and A is open (closed, bounded, compact, etc), does $f(A)$ remain open (closed, etc)?

Example 27

- Consider $f(x) = x^2$ where $f : A = (-1, 1) \mapsto \mathbb{R}$. Is $f(A)$ still open?
- Consider $f(x) = \frac{1}{x^2+1} : A = [0, \infty) \mapsto \mathbb{R}$. Is $f(A)$ still closed?

Preservation of Compact Sets

Theorem 28 (Preservation of Compact Sets)

Let $f : A \mapsto \mathbb{R}$ be continuous on A. If $K \subseteq A$ is compact, then $f(K)$ is compact.

Proof:

EVT

Theorem 29 (The Extreme Value Theorem)

Let K be compact. If $f : K \mapsto \mathbb{R}$ is continuous, then f attains a maximum and minimum value.

Proof:

Uniform Continuity

Remark 30

Recall: A function $f : A \mapsto \mathbb{R}$ is ***continuous at a point $c \in A$*** *if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - f(c)| < \epsilon$. A function f is* ***continuous on A*** *if it is continuous at every point in A.*

Definition 31

A function $f : A \mapsto \mathbb{R}$ is **uniformly continuous on A** if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

Remark 32

- *For continuity on A, we show continuity at each individual point c, so δ may be a function of c.*
- *For uniform continuity on A, the δ works simultaneously for all c.*

Sequential Criterion for Nonuniform Continuity

Remark 33

To "break" uniform continuity, it is enough to find a single ϵ so that no single δ works for all $c \in A$.

Theorem 34 (Sequential Criterion for Nonuniform Continuity)

A function $f : A \mapsto \mathbb{R}$ fails to be uniformly continuous on A if and only if there exists a particular $\epsilon > 0$ and two sequences (x_n) and (y_n) in A satisfying

$$|x_n - y_n| \xrightarrow[n\to\infty]{} 0 \text{ but } |f(x_n) - f(y_n)| \geq \epsilon.$$

Example

Example 35

The function $f(x) = \sin\left(\frac{1}{x}\right)$ is continuous on $(0, 1)$, but not uniformly continuous.

Example

Example 36

Let $h(x) = x^2$.

- h is continuous, but not uniformly continuous on $\mathbb{R}$.
- $k(x) = x^2 : [-2, 2] \mapsto \mathbb{R}$ is uniformly continuous on $[-2, 2]$.

Some Compact Theory

Theorem 37

A function that is continuous on a compact set K is uniformly continuous on K.

Proof:

Homework

Pages: 119–120
Problems: 4.4.1, 4.4.4, 4.4.6, 4.4.9, 4.4.10

Overview

Connectedness

Definition 38

- Two nonempty sets $A, B \subseteq \mathbb{R}$ are **separated** if $\bar{A} \cap B$ and $A \cap \bar{B}$ are both empty.
- A set $E \subseteq \mathbb{R}$ is **disconnected** if it can be written as $E = A \cup B$, where A and B are nonempty separated sets.
- A set that is not disconnected is called **connected**.

Example 39

- The sets $A = (1, 3)$ and $B = (3, 5)$ are separated. Hence, $E = A \cup B$ is disconnected.
- The sets $A = (1, 3]$ and $B = (3, 5)$ are not separated. Hence, $E = A \cup B$ is connected.
- The sets $A = (\infty, \sqrt{2}) \cap \mathbb{Q}$ and $B = (\sqrt{2}, \infty) \cap \mathbb{Q}$ are separated. Hence, $\mathbb{Q} = A \cup B$ is disconnected.

Connectedness Theory

Theorem 40

A set $E \subseteq \mathbb{R}$ is connected if and only if, for all nonempty disjoint sets A and B satisfying $E = A \cup B$, there always exists a convergent sequence $(x_n) \xrightarrow[n\to\infty]{} x$ with (x_n) in one of A or B and x an element of the other.

Remark 41

The above theorem states that a set is connected if and only if no matter how it is partitioned into two disjoint sets, it at least one of the sets contains a limit point of the other.

Connectedness Theory

Remark 42

The next theorem states only intervals in $\mathbb{R}$ *are connected. This includes infinite intervals.*

Theorem 43

A set $E \subseteq \mathbb{R}$ *is connected if and only if whenever* $a < c < b$ *with* $a, b \in E$*, it follows that* $c \in E$ *as well.*

Preservation of Connectedness

Theorem 44 (Preservation of Connectedness)

Let $f : A \mapsto \mathbb{R}$ be continuous. If $E \subseteq A$ is connected, then $f(E)$ is connected as well.

IVT

Theorem 45 (The Intermediate Value Theorem)

If $f : [a, b] \mapsto \mathbb{R}$ is continuous and if L is a real number satisfying $f(a) < L < f(b)$ or $f(a) > L > f(b)$, then there exists a point $c \in (a, b)$ where $f(c) = L$.

Proof:

Example

Example 46

Show that the function $f(x) = (x-1)(x-2)(x-3)$ contains at least one root in the interval $[0,4]$. (Use the IVT.)

Homework

Pages: 124

Problems: 4.5.2, 4.5.3, 4.5.7

Overview

Read me.

Overview

Read me