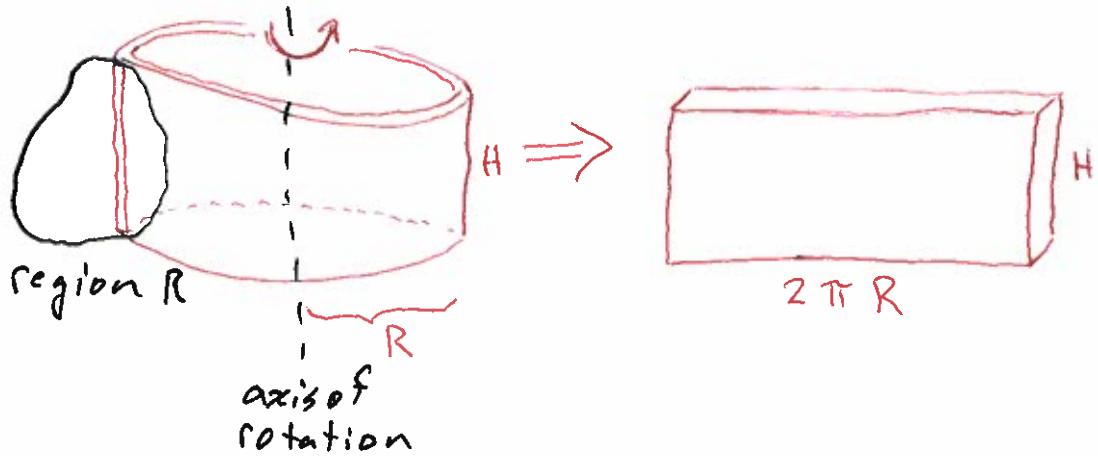


### § 6.3 Volumes by Cylindrical Shells



$$dV = 2\pi R \underbrace{H dx}_{dA}$$

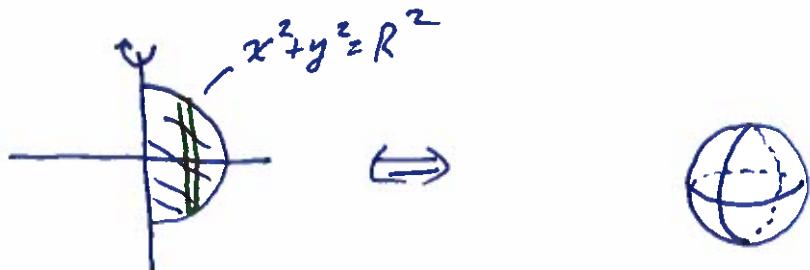
Example (i) Rotate the region bounded by  $y = \sqrt{x}$ ,  $x=2$  and  $x$ -axis, about the  $y$ -axis.



$$\left. \begin{aligned} dA &= \sqrt{x} dx \\ R &= x \end{aligned} \right\} \Rightarrow dV = 2\pi x \sqrt{x} dx$$

$$V = 2\pi \int_0^2 x^{3/2} dx = \frac{16\sqrt{2}\pi}{5}$$

Example (ii) Find the volume of a sphere of radius  $R$ .

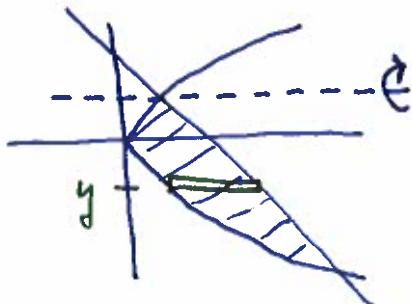


$$dA = 2\sqrt{R^2 - x^2} dx$$

$$R = \sqrt{R^2 - x^2}$$

$$\begin{aligned} V &= 4\pi \int_0^R x \sqrt{R^2 - x^2} dx & u &= R^2 - x^2 \\ &= -2\pi \int_{R^2}^0 \sqrt{u} du & du &= -2x dx \\ &= 2\pi \int_0^{R^2} u^{1/2} du \\ &= 2\pi \left( \frac{2}{3} u^{3/2} \Big|_0^{R^2} \right) \\ &= \frac{4}{3}\pi R^3 \end{aligned}$$

Example (iii) Rotate the region bounded by  $x = y^2$  and  $x + y = 2$ , about the line  $y = 1$



Intersection points:

$$y^2 + y - 2 = 0$$

$$(y+2)(y-1) = 0$$

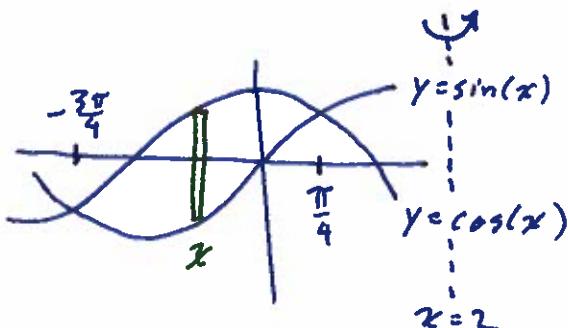
$$y = -2, 1$$

$$dA = ((2-y) - y^2) dy$$

$$R = 1-y$$

$$V = 2\pi \int_{-2}^1 (1-y)(2-y-y^2) dy = \frac{27\pi}{2}$$

Example (iv) Rotate the region bounded by  $y=\cos(x)$ ,  $y=\sin(x)$  for  $-\frac{3\pi}{4} \leq x \leq \frac{\pi}{4}$ , about the line  $x=2$ . (CAS)



$$dA = (\cos(x) - \sin(x)) dx$$

$$R = 2-x$$

$$V = 2\pi \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} (2-x)(\cos(x) - \sin(x)) dx$$

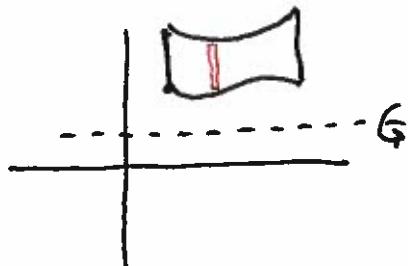
We will learn how to do this type of integral by hand in the next chapter.  
Using Mathematica we get:

$$V = \sqrt{2} \pi (8 + \pi) \approx 49,500 \text{ g}$$

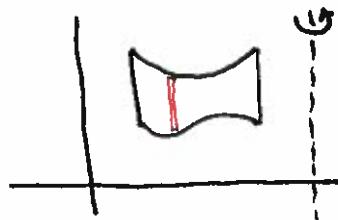
Final thoughts:



rectangle ( $dA$ ) is  
orthogonal to axis  
of rotation: Disk  
 $dV = \pi R^2 dx$



rectangle  $\perp$  axis of  
rotation; Washer  
 $dV = \pi (R^2 - r^2) dx$



rectangle parallel  
to axis of rotation:  
cylindrical shell

$$dV = 2\pi R H dx$$

The rectangles are placed at a location  $x$  (or  $y$ ). All lengths are measured "right" minus "left" or "top" minus "bottom".