§6.1 Definition of Laplace Transforms

Integral transforms

$$F(s) = \int_a^b K(s,t) f(t)\,dt$$

K is called the kernel of the transform

Laplace Transform:

$$F(s) = \mathcal{L}[f(t)](s) = \int_0^\infty e^{-st} f(t)\,dt$$

provided the limit exists

<u>Examples</u>

(i) $\mathcal{L}[c] = \int_0^\infty c\, e^{-st}\,dt$ $\qquad u = -st$, $du = -s\,dt$

$$= -\frac{c}{s}\int_0^{-\infty} e^u\,dt \text{ , provided } s>0$$

$$= -\frac{c}{s}\, e^u \Big|_0^{-\infty}$$

$$= -\frac{c}{s}(0-1)$$

$$F(s) = \frac{c}{s} \text{ , } s>0$$

(ii) $\mathcal{L}[t](s) = \int_0^\infty t e^{-st}\,dt$ integration by parts

$u = t \qquad dv = e^{-st}dt$

$du = dt \qquad v = -\frac{1}{s}e^{-st}$

$$= -\frac{t}{s}e^{-st}\Big|_0^\infty + \int_0^\infty \frac{1}{s}e^{-st}\,dt$$

$$= -\frac{1}{s}(0-0) + \left(-\frac{1}{s^2}\right)e^{-st}\Big|_0^\infty$$

$$= -\frac{1}{s^2}(0-1) \quad , \quad \text{provided } s>0$$

$$F(s) = \frac{1}{s^2} \quad , \quad s>0$$

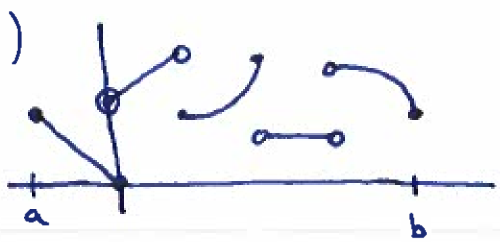
(iii) $\mathcal{L}[\sin(t)](s) = \int_0^\infty e^{-st}\sin(t)\,dt$ IBP twice, see Calc II

$$F(s) = \frac{1}{s^2+1} \quad , \quad s>0$$

<u>Def</u> f is Piecewise continuous on $[a,b]$ if there exists a finite number of points t_k such that $a = t_0 < t_1 < t_2 < \dots < t_n = b$ and f is continuous and bounded on (t_{k-1}, t_k) for $k = 1, 2, \dots, n$.

Examples

(i)



is piecewise continuous

(ii) $f(x) = \frac{1}{x-1}$

not P.C.

on $[a,b]$

where $a<1<b$

(not bounded)

(iii) $f(t) = \begin{cases} 1, & t \in \mathbb{Q} \\ 0, & t \notin \mathbb{Q} \end{cases}$

not P.C. b/c not continuous at any t value

Note: $\mathbb{Q}$ is the set of rational numbers

Theorem If f is piecewise continuous on (a,b) and $|f(t)| \leq g(t)$ for all t in (a,b), then

$$\left| \int_a^b f(t)\,dt \right| \leq \int_a^b |f(t)|\,dt \leq \int_a^b g(t)\,dt$$

Def f is of *exponential order* on $[0,\infty)$ if f is piecewise continuous on $0 \leq t \leq A$ for some $A > 0$ and $|f(t)| \leq Ke^{at}$ for $A < t < \infty$ for some values of K and a.

Examples

(i) Bounded functions are of exponential order. That is if f is piecewise continuous and there exists M such that $|f(t)| \leq M$ for all t then $|f(t)| \leq Me^{t}$ for $t > 0$.

(ii) Polynomials are of exponential order.

(iii) $f(t) = \tan(t)$ is not of exponential order b/c of the verticle asymptotes at $t = \frac{\pi}{2} + n\pi$ for n any integer ($n \in \mathbb{N}$).

(iv) $f(t) = e^{t^2}$ is not of exponential order.

Theorem If f is of exponential order (with $|f(t)| \leq k e^{at}$ for $t > A$), then

$F(s) = \mathcal{L}[f(t)](s)$ exists for $s > a$.

If f and g are of exponential order (with both $|f(t)| \leq k_1 e^{a_1 t}$ and $|g(t)| \leq k_2 e^{a_2 t}$), then

$$H(s) = \mathcal{L}[c_1 f(t) + c_2 g(t)](s) = c_1 \mathcal{L}[f(t)](s) + c_2 \mathcal{L}[g(t)](s)$$

for $s > a = \max\{a_1, a_2\}$.

That is, $\mathcal{L}$ is a linear transform.

Example

$$\mathcal{L}[2 - t + 3\sin(t)] = \frac{2}{s} - \frac{1}{s^2} + \frac{3}{s^2+1}$$