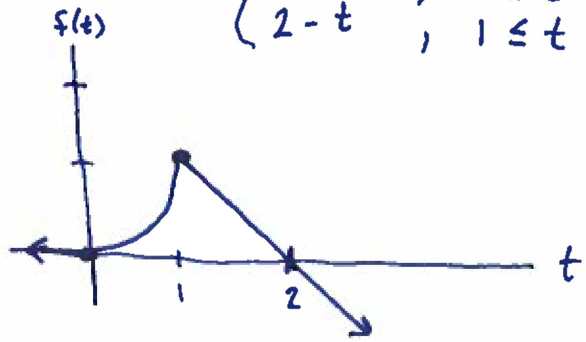
§6.1 Definition of Laplace transform
Additional examples

Example

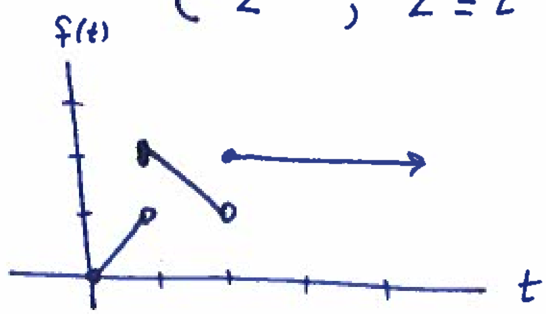
(i) Is $f(t)$ piecewise continuous?

(a) $$f(t) = \begin{cases} 0 & , t < 0 \\ t^3 & , 0 \le t < 1 \\ 2 - t & , 1 \le t \end{cases}$$



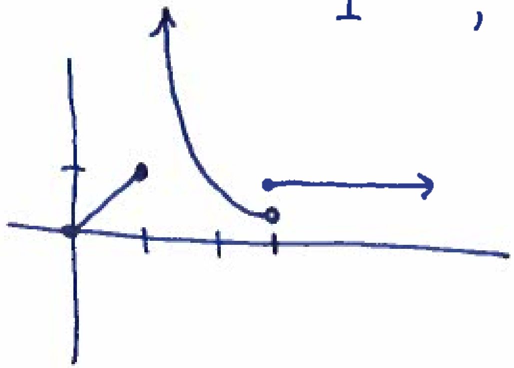
Yes. In fact it is continuous on $\mathbb{R}$

(b) $$f(t) = \begin{cases} t & , 0 \le t < 1 \\ 3 - t & , 1 \le t < 2 \\ 2 & , 2 \le t \end{cases}$$



Yes, but not continuous at $t=1$ and $t=2$.

(c) $f(t) = \begin{cases} t & , 0 \le t \le 1 \\ \frac{1}{t-1} & , 1 < t < 3 \\ 1 & , 3 \le t \end{cases}$

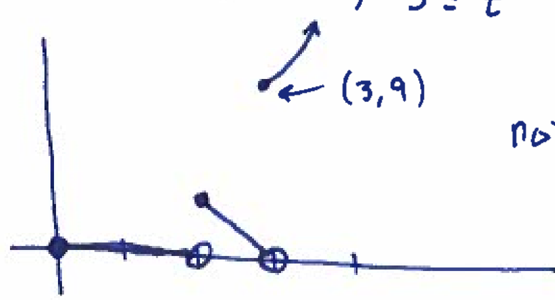


No, not bounded. Verticle asymptote at $t = 1$ $\left(\lim_{t \to 1^+} f(t) = \infty\right)$

Example

(ii) Find the k and a and A that shows $|f(t)| \le Ke^{at}$ for $t > A$ and f is piecewise continuous for $0 \le t \le A$

(a) $f(t) = \begin{cases} 0 & , 0 \le t < 2 \\ 3-t & , 2 \le t < 3 \\ t^2 & , 3 \le t \end{cases}$



not to scale

I think we can all agree that

$t < e^t$ for $t > 0^*$, so $t^2 < e^{2t}$ for $t > 0$

Thus one choice is $A = 4$, $K = 1$, $a = 2$

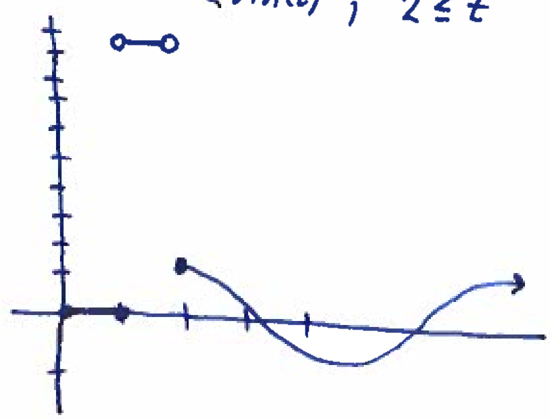
This is because f is piecewise continuous on $0 \le t \le 4$ and for $t > 4$ we have

$$|f(t)| = t^2 < 1 e^{2t}$$

* Look at a graph to see $t < e^t$

(b)

$$f(t) = \begin{cases} 0 & , \ 0 \le t \le 1 \\ 10 & , \ 1 < t < 2 \\ \sin(t) & , \ 2 \le t \end{cases}$$



not to scale

Note $|f(t)| \not< e^t$ for $1 < t < 2$, but $|\sin(t)| \le e^t$ for $t > 0$.

So choose $A = 2$ (or anything bigger), $K = 1$ and $a = \underline{1}$

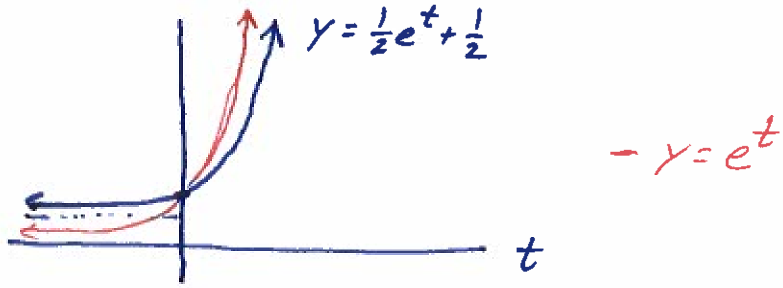
(c) $f(t) = \cosh(t)$ for $t \ge 0$.

Recall $\cosh(t) = \dfrac{e^t + e^{-t}}{2}$

for $t \ge 0, 0 < e^{-t} \le 1$

So $\cosh(t) \leq \frac{1}{2}e^t + \frac{1}{2}$ for $t \geq 0$

the graph of this is



Choose $A = 0$, $K = 1$, $a = 1$

(d) $f(t) = 4t^3 e^{3t}$, $t > 0$

Since $t < e^t$ for $t \geq 0$, $t^3 < (e^t)^3 = e^{3t}$

So $|f(t)| \leq 4e^{3t}e^{3t} = 4e^{6t}$ for $t \geq 0$

Choose $A = 0$, $K = 4$, $a = 6$.

Note that there are other choices that also work, for instance $A = 10^{10}$, $K = 1$, $a = 4$ also works (but would be harder to see graphically). $4t^3 e^{3t} < e^{4t}$ for $t > 10^{10}$

Example

(iii) Find the Laplace transform of $f(t)$

(a) $$f(t) = \begin{cases} 1 & , \ 0 \le t \le 2 \\ t^2 & , \ 2 < t \end{cases}$$

Note that where we put the equals won't matter in the integral.

$$\mathcal{L}[f(t)](s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^2 e^{-st}\, dt + \int_2^\infty t^2 e^{-st}\, dt$$

The first integral is a simple sub.:

$$\int_0^2 e^{-st}\, dt = -\frac{1}{s}\int_0^{-2s} e^u\, du \qquad \begin{array}{l} u = -st \\ du = -s\,dt \end{array}$$

$$= -\frac{1}{s}\left(e^{-2s} - e^0\right)$$

$$= \frac{1 - e^{-2s}}{s}$$

The second integral involves integration by parts twice:

$$\int_2^\infty t^2 e^{-st}\, dt = -\frac{1}{s} t^2 e^{-st}\Big|_2^\infty - \int_2^\infty \left(-\frac{2}{s}\right) t e^{-st}\, dt \qquad \begin{array}{ll} u = t^2 & dv = e^{-st}dt \\ du = 2t\,dt & v = -\frac{1}{s}e^{-st} \end{array}$$

If $s > 0$, then $\lim\limits_{t \to \infty} t^2 e^{-st} = 0$ so,

$$\int_2^\infty t^2 e^{-st}\,dt = 4\frac{e^{-2s}}{s} + \frac{2}{s}\int_2^\infty t e^{-st}\,dt \quad , \quad s>0$$

Do IBP again with $u = t$, $du = dt$, $dv = e^{-st}dt$, $v = -\frac{1}{s}e^{-st}$

to get

$$\int_2^\infty t^2 e^{-st}\,dt = 4\frac{e^{-2s}}{s} + \frac{2}{s}\left(-\frac{1}{s}te^{-st}\Big|_2^\infty - \int_2^\infty \left(-\frac{1}{s}\right)e^{-st}\,dt\right)$$

$$= 4\frac{e^{-2s}}{s} + \frac{2}{s}\left(2\frac{e^{-2s}}{s} + \frac{1}{s}\int_2^\infty e^{-st}\,dt\right)$$

$$= 4\frac{e^{-2s}}{s} + 4\frac{e^{-2s}}{s^2} + \frac{2}{s^2}\left(-\frac{1}{s}e^{-st}\right)\Big|_2^\infty$$

$$= 4\frac{e^{-2s}}{s} + 4\frac{e^{-2s}}{s^2} - 2\frac{e^{-2s}}{s^3}$$

So

$$\mathcal{L}[f(t)](s) = \left(\frac{1}{s} - \frac{e^{-2s}}{s}\right) + \left(4\frac{e^{-2s}}{s} + 4\frac{e^{-2s}}{s^2} - 2\frac{e^{-2s}}{s^3}\right)$$

$$= \frac{1}{s} + e^{-2s}\left(\frac{3}{s} + \frac{4}{s^2} - \frac{2}{s^3}\right)$$

(b) Lunch time and I'm hungry so not example (iii)(b).