§6.2 Solutions of Initial Value Problems

THE Most Important Thing In This Section Is Table 6.2.1 On Page 321!!!

Suppose $Y(s) = \mathcal{L}[y(t)](s) = \int_0^\infty e^{-st} y(t)\,dt$

then
$$\mathcal{L}[y'(t)](s) = \int_0^\infty e^{-st} y'(t)\,dt \quad \text{IBP}$$

$$u = e^{-st} \quad dv = y'(t)dt$$
$$du = -se^{-st}dt \quad v = y(t)$$

$$= e^{-st} y(t)\Big|_0^\infty + s\int_0^\infty e^{-st} y(t)\,dt$$

So, if y is of exponential order with $|y(t)| \le ke^{at}$, then

$$\mathcal{L}[y'(t)](s) = sY(s) - y(0)$$

Similarly
$$\mathcal{L}[y''(t)](s) = s^2 Y(s) - sy(0) - y'(0)$$

and
$$\mathcal{L}[y'''(t)](s) = s^3 Y(s) - s^2 y(0) - sy'(0) - y''(0)$$

Examples

(i) $\quad y' + 2y = 1 \quad , \quad y(0) = 3$

Take the Laplace transform of the DE:

$$\underbrace{sY(s) - 3 + 2Y(s)}_{\text{from previous page}} = \frac{1}{s}$$

form 1 in the table

$$(s+2)Y(s) = 3 + \frac{1}{s}$$

$$Y(s) = \frac{3}{s+2} + \frac{1}{s(s+2)}$$

Partial fraction Decomposition on the second term:

$$\frac{1}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2} \iff$$

$$1 = A(s+2) + Bs$$ by multiplying both sides with $s(s+2)$

then plug in values of s:

$$s = 0 \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$$

$$s = -2 \Rightarrow 1 = -2B \Rightarrow B = -\frac{1}{2}$$

So

$$Y(s) = \frac{3}{s+2} + \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{s+2}$$

$$Y(s) = \frac{5}{2} \cdot \frac{1}{s+2} + \frac{1}{2} \cdot \frac{1}{s}$$

Now use the table on page 321 (form 1 and 2)

$$y(t) = \frac{5}{2}e^{-2t} + \frac{1}{2}$$

(ii) $y'' + 3y' - 10y = e^{2t}$, $y(0) = 1$, $y'(0) = 0$

Transform the DE

$$s^2 Y(s) - s - 0 + 3(sY(s) - 1) - 10\,Y(s) = \frac{1}{s-2}$$

$$(s^2 + 3s - 10)\,Y(s) - s - 3 = \frac{1}{s-2}$$

$$(s+5)(s-2)\,Y(s) = s + 3 + \frac{1}{s-2}$$

$$Y(s) = \frac{s+3}{(s+5)(s-2)} + \frac{1}{(s+5)(s-2)^2}$$

Using Apart command in Mathematica to do the Partial Fraction Decomposition:

$$Y(s) = \frac{15}{49}\cdot\frac{1}{s+5} + \frac{34}{49}\cdot\frac{1}{s-2} + \frac{1}{7}\cdot\frac{1}{(s-2)^2}$$

then use #2 and #11 of the table

$$y(t) = \frac{15}{49}e^{-5t} + \frac{34}{49}e^{2t} + \frac{1}{7}te^{2t}$$

Note that #3 in the table says

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}} \quad , \quad s > 0$$

the proof of this involves IBP n times:

$$\mathcal{L}[t^n] = \int_0^\infty t^n e^{-st}\,dt$$

Note # 14 :

$$\mathcal{L}\left[e^{at} f(t)\right](s) = \int_0^\infty e^{-st} e^{at} f(t)\, dt$$
$$= \int_0^\infty e^{-(s-a)t} f(t)\, dt$$
$$= F(s-a)$$

Example

(i) Since $\mathcal{L}\left[\sin(bt)\right](s) = \dfrac{b}{s^2+b^2}$ from # 5

then $\mathcal{L}\left[e^{at}\sin(bt)\right](s) = \dfrac{b}{(s-a)^2+b^2}$ this is # 9

(ii) Since $\mathcal{L}\left[t^n\right](s) = \dfrac{n!}{s^{n+1}}$ from # 3

then $\mathcal{L}\left[t^n e^{at}\right](s) = \dfrac{n!}{(s-a)^{n+1}}$ this is # 11

(iii) $$\mathcal{L}^{-1}\left[\frac{1}{(s-4)^5}\right](t) = e^{4t}\mathcal{L}^{-1}\left[\frac{1}{s^5}\right]$$
$$= \left(e^{4t}\right)\left(\frac{1}{4!}\cdot t^4\right)$$
$$= \frac{1}{24} t^4 e^{4t}$$

(iv) $$\mathcal{L}^{-1}\left[\frac{(s+2)}{(s+2)^2-9}\right](t) = e^{-2t}\cosh(3t)$$