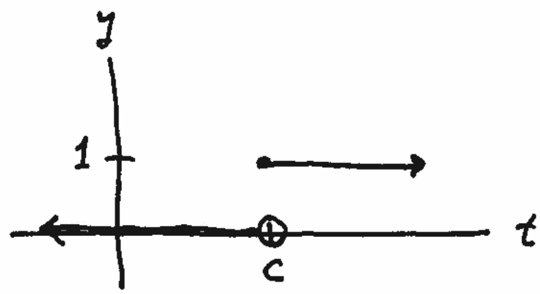
§6.3 Step Functions

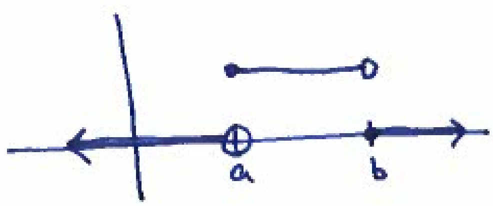
The unit step function or Heaviside function:

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases} \quad \text{where } c \geq 0$$

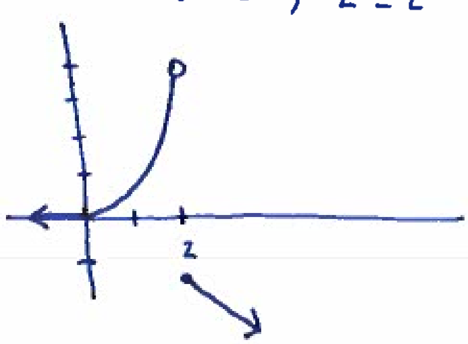


Example

(i) If $a < b$, $f(t) = u_a(t) - u_b(t) = \begin{cases} 0, & t < a \\ 1, & a \leq t < b \\ 0, & t \geq b \end{cases}$



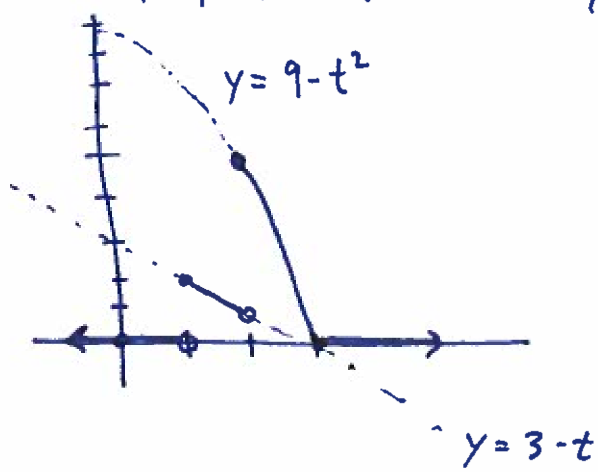
(ii) If $f(t) = \begin{cases} 0, & t < 0 \\ t^2, & 0 \leq t < 2 \\ 1-t, & 2 \leq t \end{cases} = t^2 u_0(t) + (1-t-t^2)u_2(t)$



(iii) $$f(t) = \begin{cases} 0, & t<1 \\ 3-t, & 1 \le t < 2 \\ 9-t^2, & 2 \le t < 3 \\ 0, & 3 \le t \end{cases}$$

$$= \underbrace{(3-t)u_1(t)}_{\substack{\text{turn on} \\ 3-t \text{ at } t=1}} + \underbrace{((9-t^2)-(3-t))u_2(t)}_{\substack{\text{turn on } 9-t^2 \text{ and} \\ \text{turn off } 3-t \text{ at } t=1}} - \underbrace{(9-t^2)u_3(t)}_{\substack{\text{turn off } 9-t^2 \\ \text{at } t=3}}$$

$$= (3-t)u_1(t) + (6-t-t^2)u_2(t) + (t^2-9)u_3(t)$$



The Laplace transform for the Heaviside function:

$$\mathcal{L}[u_c(t)](s) = \int_0^\infty e^{-st} u_c(t)\,dt$$

$$= \int_0^c 0\,dt + \int_c^\infty e^{-st}\,dt$$

$$= -\frac{1}{s} e^{-st} \Big|_c^\infty$$

$$= \frac{e^{-cs}}{s}$$

This is #12 on Table 6.2.1, page 321.
Also look at #13 and #14.

Examples

(i) $$\mathcal{L}\left[(t-2)^4 u_2(t)\right](s) \overset{\#13}{=} e^{-2s}\,\mathcal{L}\left[t^4\right](s)$$

$$\overset{\#4}{=} e^{-2s} \cdot \frac{4!}{s^5}$$

(ii) $$\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s^2+4}\right](t) \overset{\#13}{=} u_3(t)\,\mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right](t-3)$$

$$\overset{\#5}{=} u_3(t)\left(\frac{1}{2}\cdot \sin(2(t-3))\right)$$

$$= \frac{1}{2}\sin(2t-6)\,u_3(t)$$

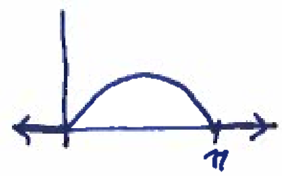
Note $\mathcal{L}\left[\frac{2}{s^2+4}\right](t) \overset{\#5}{=} \sin(2t)$

So the $\frac{1}{2}$ in the answer compensates for the missing 2 in $\mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right](t-3)$ and we evaluate the $\sin(2u)$ at $u=t-3$.

The following example demonstrates how to solve on ODE with a "Forcing Function" (nonhomogeneous). This is covered in section 6.4 of the book. The remainder of chapter 6 is good material that should be looked at, but because of time constraints we will move on to chapter 7.

Example

Let $f(t) = \begin{cases} 0, & t < 0 \\ \sin(t), & 0 \le t \le \pi \\ 0, & t > \pi \end{cases}$



Solve the IVP:

$$y'' + 3y' + 2y = f(t), \quad y(0) = 1, \quad y'(0) = -3.$$

First convert f into Heaviside:

$$f(t) = \sin(t)u_0(t) - \sin(t)u_\pi(t).$$

Since #13 in Table 6.2.1 says

$$\mathcal{L}[u_c(t)f(t-c)](s) = e^{-cs}\mathcal{L}[f(t)](s)$$

and $\sin(t) = -\sin(t-\pi)$ we rewrite f as

$$f(t) = \sin(t)u_0(t) + \sin(t-\pi)u_\pi(t).$$

Now take the Laplace transform of the D.E using #18, #13, and #5 from the Table and let $Y(s) = \mathcal{L}[y(t)](s)$:

$$\mathcal{L}[y''(t)] + 3\mathcal{L}[y'(t)] + 2\mathcal{L}[y(t)] = \mathcal{L}[f(t)]$$

$$\Longrightarrow \quad s^2 Y(s) - s\,y(0) - y'(0) + 3\big(s\,Y(s) - y(0)\big) + 2\,Y(s) =$$

$$e^{-0s} \cdot \frac{1}{s^2+1} + e^{-\pi s} \cdot \frac{1}{s^2+1}$$

$$\Longrightarrow \quad (s^2 + 3s + 2)Y(s) - s + 3 + 3(-1) = \left(1 + e^{-\pi s}\right) \cdot \frac{1}{s^2+1}$$

$$\Longrightarrow \quad (s+1)(s+2)Y(s) = s + \left(1 + e^{-\pi s}\right)\frac{1}{s^2+1}$$

$$\Longrightarrow \quad Y(s) = \frac{s}{(s+1)(s+2)} + \left(1 + e^{-\pi s}\right) \cdot \frac{1}{(s+1)(s+2)(s^2+1)}$$

Now do a partial fraction decomposition:

(a) $$\frac{s}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

(b) $$\frac{1}{(s+1)(s+2)(s^2+1)} = \frac{C}{s+1} + \frac{D}{s+2} + \frac{Ex+F}{s^2+1}$$

We would then use calc II (really algebra) to find A, B, C, D, E, F. I will use the Apart command in Mathematica:

$$A = -1, \quad B = 2, \quad C = \frac{1}{2}, \quad D = -\frac{1}{5}, \quad E = -\frac{3}{10}, \quad F = \frac{1}{10}$$

Mathematica code on the next page.

In[•]:= a = Apart[s / ((s + 1) (s + 2)), s]

Out[•]= $-\frac{1}{1+s}+\frac{2}{2+s}$

In[•]:= b = Apart[1 / ((s + 1) (s + 2) (s^2 + 1)), s]

Out[•]= $\frac{1}{2(1+s)}-\frac{1}{5(2+s)}+\frac{1-3s}{10(1+s^2)}$

In[•]:= a + b

Out[•]= $-\frac{1}{2(1+s)}+\frac{9}{5(2+s)}+\frac{1-3s}{10(1+s^2)}$

So $$Y(s)=\frac{2}{s+2}-\frac{1}{s+1}+(1+e^{-\pi s})\left(\frac{1}{2}\cdot\frac{1}{s+1}-\frac{1}{5}\cdot\frac{1}{s+2}+\frac{1}{10}\cdot\frac{1}{s^2+1}-\frac{3}{10}\cdot\frac{s}{s^2+1}\right)$$

Combine some like terms (that is what a+b was for above)

$$Y(s)=\frac{9}{5}\cdot\frac{1}{s+2}-\frac{1}{2}\cdot\frac{1}{s+1}+\frac{1}{10}\frac{1}{s^2+1}-\frac{3}{10}\cdot\frac{s}{s^2+1}$$
$$+e^{-\pi s}\left(\frac{1}{2}\cdot\frac{1}{s+1}-\frac{1}{5}\cdot\frac{1}{s+2}+\frac{1}{10}\cdot\frac{1}{s^2+1}-\frac{3}{10}\cdot\frac{s}{s^2+1}\right)$$

Now use the table to take the inverse Laplace transform. Here are the components we need

$\mathcal{L}^{-1}\left[\frac{1}{s+1}\right](t)=e^{-t}$ by # 2

$\mathcal{L}^{-1}\left[\frac{1}{s+2}\right](t)=e^{-2t}$ by # 2

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right](t) = \sin(t) \quad \text{by } \# 5$$

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right](t) = \cos(t) \quad \text{by } \# 6$$

and we will also use # 13. Don't forget that $\mathcal{L}^{-1}[Y(s)](t) = y(t)$, so the solution to the IVP is:

$$y(t) = \frac{9}{5}e^{-2t} - \frac{1}{2}e^{-t} + \frac{1}{10}\sin(t) - \frac{3}{10}\cos(t)$$
$$+ u_\pi(t)\left(\frac{1}{2}e^{-(t-\pi)} - \frac{1}{5}e^{-2(t-\pi)} + \frac{1}{10}\sin(t-\pi) - \frac{3}{10}\cos(t-\pi)\right)$$

Wasn't that fun!

Note: Any polynomial can be shifted:

Let $f(t) = t^3$ and write it as a power series centered at 2.

$$f(t) = t^3 \Rightarrow f(2) = 8$$
$$f'(t) = 3t^2 \Rightarrow f'(2) = 12$$
$$f''(t) = 6t \Rightarrow f''(2) = 12$$
$$f'''(t) = 6 \Rightarrow f'''(2) = 6$$
$$f^{(n)}(t) = 0 \text{ for } n>3 \Rightarrow f^{(n)}(2) = 0$$

so $t^3 = 8 + 12(t-2) + \frac{12}{2!}(t-2)^2 + \frac{6}{3!}(t-2)^3$