

## §7.1 Integration By Parts

Recall the product rule for derivatives  
(written in differential form)

$$d(uv) = u dv + v du$$

Integrate to get

$$uv = \int u dv + \int v du$$

so

$$\boxed{\int u dv = uv - \int v du}$$

$$\int_a^b u dv = uv \Big|_a^b - \int_a^b v du$$

Example (i)

$$\begin{aligned} & \int_1^2 x \ln(x) dx && u = \ln(x) \quad dv = x dx \\ &= (\ln(x)) \left( \frac{1}{2} x^2 \right) \Big|_1^2 - \int_1^2 \left( \frac{1}{2} x^2 \right) \left( \frac{1}{x} \right) dx && du = \frac{1}{x} dx \quad v = \frac{1}{2} x^2 \\ &= 2 \ln(2) - \frac{1}{2} \ln(1) - \frac{1}{2} \int_1^2 x dx && \text{(note: no } +c\text{)} \\ &= 2 \ln(2) - \frac{1}{2} \left( \frac{1}{2} x^2 \right) \Big|_1^2 \\ &= \ln(4) - \frac{1}{4} (4 - 1) \\ &= \ln(4) - \frac{3}{4} \end{aligned}$$

## Example (ii)

$$\begin{aligned}
 & \int x^2 \cos(x) dx \\
 &= x^2 \sin(x) - \int 2x \sin(x) dx \quad \text{IBP again} \\
 &= x^2 \sin(x) - (-2x \cos(x) - \int (-2 \cos(x)) dx) \\
 &= x^2 \sin(x) + 2x \cos(x) - 2 \int \cos(x) dx \\
 &= x^2 \sin(x) + 2x \cos(x) - 2 \sin(x) + C
 \end{aligned}$$

Note: the multiple IBP can be summarized in a tabular form:

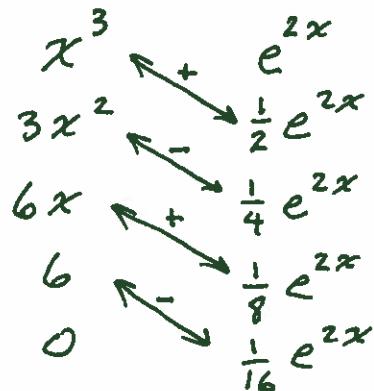
$x^2$ $2x$ $2$ $0$	$\cos(x)$ $\sin(x)$ $-\cos(x)$ $-\sin(x)$	Original $u = x^2, dv = \cos(x) dx$ . Differentiate the first column until zero, Integrate the second column. Products have $+, -, +, -, \dots$ .
-----------------------------	--	---

$$\Rightarrow x^2 \sin(x) + 2x \cos(x) - 2 \sin(x) + C$$

### Example (iii)

$$I = \int x^3 e^{2x} dx$$

This would require IBP 3 times with original ~~8~~  $u = x^3$  and  $dv = e^{2x} dx$ . Using Tabular IBP:



$$\begin{aligned} I &= \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{6}{8}x e^{2x} - \frac{6}{16} e^{2x} + C \\ &= \left( \frac{1}{2}x^3 - \frac{3}{4}x^2 + \frac{3}{4}x - \frac{3}{8} \right) e^{2x} + C \end{aligned}$$

### Example (iv)

$$\begin{aligned} &\int_0^1 \tan^{-1}(x) dx \quad u = \tan^{-1}(x) \quad dv = dx \\ &\quad du = \frac{1}{1+x^2} dx \quad v = x \\ &= x \tan^{-1}(x) \Big|_0^1 - \int_0^1 \underbrace{\frac{x}{1+x^2} dx}_{\text{simple sub.}} \quad u = 1+x^2 \quad du = 2x dx \\ &= \frac{\pi}{4} - \frac{1}{2} \int_1^2 \frac{1}{u} du = \frac{\pi}{4} - \frac{1}{2} \ln(2) \end{aligned}$$

### Example (v)

$$I = \int e^{ax} \cos(bx) dx \quad \text{where } a, b \text{ are constants.}$$

so

$$\begin{aligned} u &= e^{ax} & dv &= \cos(bx) dx \\ du &= a e^{ax} dx & v &= \frac{1}{b} \sin(bx) \end{aligned}$$

$$I = \frac{1}{b} e^{ax} \sin(bx) - \frac{a}{b} \int e^{ax} \sin(bx) dx$$

IBP again using the same pattern

so

$$\begin{aligned} u &= e^{ax} & dv &= \sin(bx) dx \\ du &= a e^{ax} dx & v &= -\frac{1}{b} \cos(bx) \end{aligned}$$

$$I = \frac{1}{b} e^{ax} \sin(bx) - \frac{a}{b} \left( -\frac{1}{b} e^{ax} \cos(bx) + \frac{a}{b} \int e^{ax} \cos(bx) dx \right)$$

that is

$$I = \frac{1}{b} e^{ax} \sin(bx) + \frac{a}{b^2} e^{ax} \cos(bx) - \frac{a^2}{b^2} I$$

$$\Leftrightarrow \left(1 + \frac{a^2}{b^2}\right) I = \frac{e^{ax}}{b^2} (b \sin(bx) + a \cos(bx))$$

so

$$I = \frac{e^{ax}}{a^2 + b^2} (b \sin(bx) + a \cos(bx))$$

and

$$\int e^{ax} \cos(bx) dx = \frac{e^{ax}}{a^2 + b^2} (b \sin(bx) + a \cos(bx)) + C$$

Example (vi)

$$I = \int (\ln(x))^3 dx$$

Use a simple substitution  $t = \ln(x) \Leftrightarrow$   
 $x = e^t, dx = e^t dt.$  so

$$I = \int t^3 e^t dt$$

Then tabular IBP

$$\begin{array}{ccc} t^3 & \xrightarrow{+} & e^t \\ 3t^2 & \xrightarrow{-} & e^t \\ 6t & \xrightarrow{+} & e^t \\ 6 & \xrightarrow{-} & e^t \\ 0 & \xrightarrow{+} & e^t \end{array}$$

so

$$I = t^3 e^t - 3t^2 e^t + 6t e^t - 6e^t + C$$

$$\int (\ln(x))^3 dx = x (\ln(x)^3 - 3\ln(x)^2 + 6\ln(x) - 6) + C$$