

## §7.2 Trigonometric Integrals

We know

$$\int \sin(x) dx = -\cos(x) + C$$

$$\int \cos(x) dx = \sin(x) + C$$

$$\int \sec^2(x) dx = \tan(x) + C$$

⋮

But...

### Example (i)

$$\int \tan(x) dx$$

$$= \int \frac{\sin(x)}{\cos(x)} dx$$

$$= - \int \frac{1}{u} du$$

$$= -\ln|u| + C$$

$$= \ln|\sec(x)| + C$$

simple substitution

$$u = \cos(x), \quad du = -\sin(x) dx$$

Note  $-\ln|\cos(x)| = \ln\left|\frac{1}{\cos(x)}\right|$   
by properties of logarithms.

### Example (ii)

$$\int \sec(x) dx$$

simple sub. (sort of 😊)

$$= \int \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x) + \tan(x)} dx$$

$$\text{so } u = \sec(x) + \tan(x)$$

$$du = (\sec(x)\tan(x) + \sec^2(x)) dx$$

$$= \int \frac{1}{u} du$$

$$= \ln|u| + C$$

$$= \ln|\sec(x) + \tan(x)| + C$$

The integrals for  $\cot(x)$  and  $\csc(x)$  are similar to these examples above. So now we "know" the antiderivatives for all six trig. functions. Next some other fun tricks with trig. integrals:

### Example (iii)

$$\int \sin^3(x) \cos^4(x) dx \quad \text{odd power on sin} \Rightarrow$$
$$u = \cos(x), \quad du = -\sin(x)$$

$$= \int \sin^2(x) \cos^4(x) \sin(x) dx$$

$$= \int (1 - \cos^2(x)) (\cos^4(x)) \sin(x) dx$$

$$= - \int (1 - u^2) (u^4) du$$

$$= \int (u^6 - u^4) du$$

$$= \frac{1}{7} u^7 - \frac{1}{5} u^5 + C$$

$$= \frac{1}{7} \cos^7(x) - \frac{1}{5} \cos^5(x) + C$$

Note: odd power on cos  $\Rightarrow$   
 $u = \sin(x)$   
So simple sub. with some algebra and  $\cos^2(\theta) + \sin^2(\theta) = 1$  identity.

### Example (iv)

$$\int \sin^2(t) \cos^4(t) dt$$

Even powers on sin and cos  $\Rightarrow$  double angle identities for cos.

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$$\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2} \quad \text{and} \quad \cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}$$

$$\text{Note: } \cos(2\theta) = \begin{cases} \cos^2(\theta) - \sin^2(\theta) \\ 2\cos^2(\theta) - 1 \\ 1 - 2\sin^2(\theta) \end{cases}$$

from trigonometry or Pre-calculus.

So the integral becomes

$$\begin{aligned} & \int \frac{1 - \cos(2t)}{2} \left( \frac{1 + \cos(2t)}{2} \right)^2 dt \quad \text{mult. out} \\ &= \frac{1}{8} \int (1 + \cos(2t) - \underbrace{\cos^2(2t)}_{\substack{\text{even power} \\ \text{so * again}}} - \underbrace{\cos^3(2t)}_{\substack{\text{odd power} \\ \text{so } u = \sin(2t), du = 2\cos(2t)dt}}) dt \\ &= \frac{1}{8} \left( t + \frac{1}{2} \sin(2t) - \int \frac{1 + \cos(4t)}{2} dt - \frac{1}{2} \int (1 - u^2) du \right) \\ &= \frac{1}{8} \left( t + \frac{1}{2} \sin(2t) - \frac{1}{2} \left( t + \frac{1}{4} \sin(4t) \right) - \frac{1}{2} \left( \sin(2t) - \frac{1}{3} \sin^3(2t) \right) \right) \\ & \quad + C \end{aligned}$$

then combine like terms.

Note that we convert the integral into several "easier" integrals, then work each new integral as a separate problem.

### Example (v)

$$I = \int \sec^n(x) dx$$

Here we use integration by parts to get a reduction formula.

$$\left. \begin{aligned} u &= \sec^{n-2}(x) \\ du &= (n-2) \sec^{n-3}(x) \cdot \sec(x) \tan(x) dx \\ &= (n-2) \sec^{n-2}(x) \tan(x) dx \\ dv &= \sec^2(x) dx \\ v &= \tan(x) \end{aligned} \right\} \Rightarrow$$

$$I = \sec^{n-2}(x) \tan(x) - \int (n-2) \sec^{n-2}(x) \tan^2(x) dx$$

$$\text{Recall } \cos^2(\theta) + \sin^2(\theta) = 1 \Leftrightarrow 1 + \tan^2(\theta) = \sec^2(\theta)$$

So

$$I = \sec^{n-2}(x) \tan(x) - (n-2) \int \sec^{n-2}(x) (\sec^2(x) - 1) dx$$

$$I = \sec^{n-2}(x) \tan(x) - (n-2)I + (n-2) \int \sec^{n-2}(x) dx$$

Solve for I

$$I = \frac{1}{n-1} \sec^{n-2}(x) \tan(x) + \frac{n-2}{n-1} \int \sec^{n-2}(x) dx$$

### Example (vb)

$$\int \sec^4(x) dx$$

Use the previous reduction formula with  $n=4$

$$= \frac{1}{3} \sec^2(x) \tan(x) - \frac{2}{3} \int \sec^2(x) dx$$

$$= \frac{1}{3} \sec^2(x) \tan(x) - \frac{2}{3} \tan(x) + C$$

### Example (vc)

$$\int \sec^3(x) \tan^4(x) dx$$

$$= \int \sec^3(x) (\sec^2(x) - 1)^2 dx$$

$$= \int (\sec^7(x) - 2 \sec^5(x) + \sec^3(x)) dx$$

then use the reduction formula on each of these integrals.

$$\int \sec^7(x) dx = \frac{1}{6} \sec^5(x) \tan(x) + \frac{5}{6} \int \sec^5(x) dx \Rightarrow$$

$$= \frac{1}{6} \sec^3(x) \tan(x) - \frac{7}{6} \int \sec^3(x) dx + \int \sec^3(x) dx$$

$$\text{But } \int \sec^5(x) dx = \frac{1}{4} \sec^3(x) \tan(x) + \frac{3}{4} \int \sec^3(x) dx$$

$$= \dots$$

Note: Mathematica may use different techniques so it may get different "looking" results.

## Example (vi)

$$\int \sec^4(x) \tan^3(x) dx$$

This one is a simple sub. In fact, you can't go wrong on this one.

If  $u = \sec(x)$ ,  $du = \sec(x) \tan(x) dx$  and

$$= \int \underbrace{\sec^3(x)}_{u^3} \underbrace{\tan^2(x)}_{u^2-1} \underbrace{\sec(x) \tan(x) dx}_{du}$$

$$= \int u^3(u^2-1) du$$

$$= \frac{1}{6} \sec^6(x) - \frac{1}{4} \sec^4(x) + C$$

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If instead you let  $u = \tan(x)$ ,  $du = \sec^2(x) dx$  and

$$= \int \underbrace{\sec^2(x)}_{1+u^2} \underbrace{\tan^3(x)}_{u^3} \underbrace{\sec^2(x) dx}_{du}$$

$$= \int (1+u^2) u^3 du$$

⋮

$$= \frac{1}{4} \tan^4(x) + \frac{1}{6} \tan^6(x) + C$$

Note: these are the same b/c  $1 + \tan^2(\theta) = \sec^2(\theta)$  and the arbitrary constant.

Final thoughts these are only some of the common types of integrals that involve trig. functions. Most of this section is a reminder that trig. identities exist and are now useful.

Recall:  $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{1}{\cot(\theta)}$

$$\sec(\theta) = \frac{1}{\cos(\theta)} \text{ and } \csc(\theta) = \frac{1}{\sin(\theta)}$$

Also look at the "Strategy" boxes in the section and Table 2 on page 476.

The overall goal is to manipulate the integral into one where a simple substitution may be used or IBP.