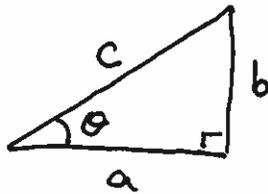


## §7.3 Trigonometric Substitution



$$c^2 = a^2 + b^2$$

$$c = \sqrt{a^2 + b^2}$$

Similarly

$$a = \sqrt{c^2 - b^2}$$

$$b = \sqrt{c^2 - a^2}$$

$$\cos(\theta) = \frac{a}{c} \Rightarrow \begin{cases} a = c \cos(\theta) \\ c = a \sec(\theta) \end{cases}$$

$$\sin(\theta) = \frac{b}{c} \Rightarrow b = c \sin(\theta)$$

$$\tan(\theta) = \frac{b}{a} \Rightarrow b = a \tan(\theta)$$

The substitutions we will look for:

$$u = a \tan(\theta)$$

$$u = a \sec(\theta)$$

$$du = a \sec^2(\theta) d\theta \quad \text{-or-} \quad du = a \sec(\theta) \tan(\theta) d\theta$$

$$\sqrt{u^2 + a^2} = a \sec(\theta)$$

$$\sqrt{u^2 - a^2} = a \tan(\theta)$$

$$\text{-or-} \quad u = a \sin(\theta)$$

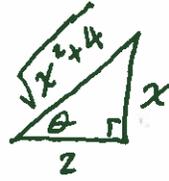
$$du = a \cos(\theta) d\theta$$

$$\sqrt{a^2 - u^2} = a \cos(\theta)$$

We look for  $u^2 + a^2$ ,  $u^2 - a^2$ , or  $a^2 - u^2$  in the integrals

### Example (i)

$$\int \frac{1}{\sqrt{x^2+4}} dx$$



$$x = 2 \tan(\theta)$$
$$dx = 2 \sec^2(\theta) d\theta$$

$$\sqrt{x^2+4} = 2 \sec(\theta)$$

$$= \int \frac{2 \sec^2(\theta)}{2 \sec(\theta)} d\theta$$

$$= \int \sec(\theta) d\theta$$

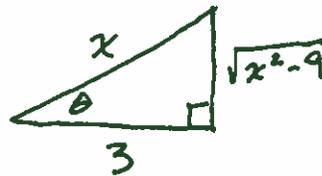
$$= \ln |\sec(\theta) + \tan(\theta)| + C \quad \text{then back sub.}$$

$$= \ln \left| \frac{\sqrt{x^2+4}}{2} + \frac{x}{2} \right| + C$$

$$= \ln |\sqrt{x^2+4} + x| + C \quad \text{by properties of logarithms and } C \text{ is arbitrary.}$$

### Example (ii)

$$\int \frac{1}{\sqrt{x^2-9}} dx$$



$$x = 3 \sec(\theta)$$

$$dx = 3 \sec(\theta) \tan(\theta) d\theta$$

$$\sqrt{x^2-9} = 3 \tan(\theta)$$

$$= \int \frac{3 \sec(\theta) \tan(\theta)}{3 \tan(\theta)} d\theta$$

$$= \int \sec(\theta) d\theta$$

$$= \ln |\sec(\theta) + \tan(\theta)| + C$$

$$= \ln \left| \frac{x}{3} + \frac{\sqrt{x^2-9}}{3} \right| + C$$

$$= \ln |x + \sqrt{x^2-9}| + C$$

### Example (iii)

$$\begin{aligned} & \int \frac{1}{\sqrt{a^2 - x^2}} dx && \begin{array}{c} a \\ \theta \quad r \\ \sqrt{a^2 - x^2} \end{array} && \begin{array}{l} x = a \sin(\theta) \\ dx = a \cos(\theta) d\theta \\ \sqrt{a^2 - x^2} = a \cos(\theta) \end{array} \\ &= \int \frac{a \cos(\theta)}{a \cos(\theta)} d\theta \\ &= \int 1 d\theta \\ &= \theta + C \\ &= \sin^{-1}\left(\frac{x}{a}\right) + C \end{aligned}$$

Similarly  $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$

$$\int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right) + C$$

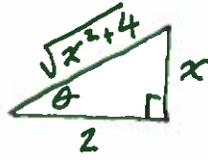
### Example (iv)

$$\begin{aligned} & \int \frac{x}{\sqrt{x^2 + 1}} dx && \begin{array}{l} \text{Simple sub:} \\ u = x^2 + 1, du = 2x dx \end{array} \\ &= \frac{1}{2} \int \frac{1}{\sqrt{u}} du \\ &= \frac{1}{2} (2u^{1/2}) + C \\ &= \sqrt{x^2 + 1} + C \end{aligned}$$

Not all problems require more complicated techniques. Look for "simple" sub. first.

### Example (v)

$$\int_0^1 x^2 \sqrt{x^2+4} dx$$



$$x = 2 \tan(\theta) \\ dx = 2 \sec^2(\theta) d\theta$$

$$\sqrt{x^2+4} = 2 \sec(\theta)$$

$$= \int_0^1 (2 \tan(\theta))^2 (2 \sec(\theta)) (2 \sec^2(\theta)) d\theta$$

$$= 16 \int_0^1 \tan^2(\theta) \sec^3(\theta) d\theta$$

$$= 16 \int_0^1 (\sec^2(\theta) - 1) \sec^3(\theta) d\theta$$

$$= 16 \int_0^1 (\sec^5(\theta) - \sec^3(\theta)) d\theta$$

$$\int \sec^5(\theta) d\theta = \frac{1}{4} \sec^3(\theta) \tan(\theta) + \frac{3}{4} \int \sec^3(\theta) d\theta$$

by the reduction formula from ~~7.2~~ 7.2

(also formula 77 on reference page 9).

$$= 16 \left( \frac{1}{4} \sec^3(\theta) \tan(\theta) \Big|_{x=0}^{x=1} - \frac{1}{4} \int_0^1 \sec^3(\theta) d\theta \right)$$

Use reduction formula again

$$= 16 \left( \frac{1}{4} \sec^3(\theta) \tan(\theta) \Big|_{x=0}^{x=1} - \frac{1}{4} \left( \frac{1}{2} \sec(\theta) \tan(\theta) \Big|_{x=0}^{x=1} + \frac{1}{2} \int_0^1 \sec(\theta) d\theta \right) \right)$$

$$= \left( 4 \sec^3(\theta) \tan(\theta) - 2 \sec(\theta) \tan(\theta) + 2 \ln |\sec(\theta) + \tan(\theta)| \right) \Big|_{x=0}^{x=1}$$

$$= \left( 4 \left( \frac{\sqrt{x^2+4}}{2} \right)^3 \left( \frac{x}{2} \right) - 2 \left( \frac{\sqrt{x^2+4}}{2} \right) \left( \frac{x}{2} \right) + 2 \ln \left| \frac{\sqrt{x^2+4}}{2} + \frac{x}{2} \right| \right) \Big|_0^1$$

$$= \left( \sqrt{5}^3 - \frac{\sqrt{5}}{2} + 2 \ln \left| \frac{\sqrt{5}}{2} + \frac{1}{2} \right| \right) - \left( 0 - 0 + 2 \ln |1+0| \right)$$

$$= \frac{9}{2} \sqrt{5} + 2 \ln \left( \frac{1+\sqrt{5}}{2} \right)$$

### Example (vi)

$$\int \frac{1}{\sqrt{x^2 - 6x + 13}} dx$$

$$\begin{aligned} \text{First: } x^2 - 6x + 13 &= x^2 - 6x + (-3)^2 + 13 - (-3)^2 \\ &= (x-3)^2 + 4 \end{aligned}$$

$$= \int \frac{1}{\sqrt{(x-3)^2 + 4}} dx \quad \text{so } u = x-3, du = dx$$

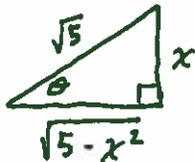
$$= \int \frac{1}{\sqrt{u^2 + 4}} dx$$

$$= \ln |\sqrt{u^2 + 4} + u| + C \quad \text{by example (i)}$$

$$= \ln |\sqrt{x^2 - 6x + 13} + x - 3| + C$$

### Example (vii)

$$\int \frac{1}{x\sqrt{5-x^2}} dx$$



$$x = \sqrt{5} \sin(\theta)$$

$$dx = \sqrt{5} \cos(\theta) d\theta$$

$$\sqrt{5-x^2} = \sqrt{5} \cos(\theta)$$

$$= \int \frac{\sqrt{5} \cos(\theta)}{(\sqrt{5} \sin(\theta))(\sqrt{5} \cos(\theta))} d\theta$$

$$= \frac{1}{\sqrt{5}} \int \frac{1}{\sin(\theta)} d\theta$$

$$= \frac{1}{\sqrt{5}} \int \csc(\theta) d\theta$$

$$= \frac{1}{\sqrt{5}} \ln |\csc(\theta) - \cot(\theta)| + C$$

$$= \frac{1}{\sqrt{5}} \ln \left| \frac{\sqrt{5}}{x} - \frac{\sqrt{5-x^2}}{x} \right| + C$$

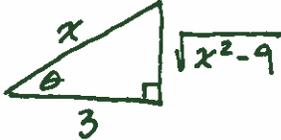
### Example (viii)

$$\begin{aligned} & \int \frac{x^3}{\sqrt{9-x^2}} dx && u = 9-x^2 \Leftrightarrow x = 9-u \\ & && du = -2x dx \\ & = \int \frac{x^2}{\sqrt{9-x^2}} \cdot x dx \\ & = -\frac{1}{2} \int \frac{9-u}{\sqrt{u}} du \\ & = \frac{1}{2} \int (u^{1/2} - 9u^{-1/2}) du \\ & = \frac{1}{2} \left( \frac{2}{3} u^{3/2} - 9 \cdot 2u^{1/2} \right) + C \\ & = \frac{1}{3} (9-x^2)^{3/2} - 9(9-x^2)^{1/2} + C \\ & = \frac{1}{3} \sqrt{9-x^2} (9-x^2-9) + C \\ & = -\frac{1}{3} x^2 \sqrt{9-x^2} + C \end{aligned}$$

### Example (ix)

$$\begin{aligned} & \int \frac{x^2}{\sqrt{x^2-9}} dx && \begin{array}{l} \text{Diagram: A right-angled triangle with hypotenuse } x, \text{ base } 3, \text{ and height } \sqrt{x^2-9}. \text{ The angle } \theta \text{ is at the bottom-left corner.} \\ x = 3 \sec(\theta) \\ dx = 3 \sec(\theta) \tan(\theta) d\theta \\ \sqrt{x^2-9} = 3 \tan(\theta) \end{array} \\ & = \int \frac{(3 \sec(\theta))^2 (3 \sec(\theta) \tan(\theta))}{3 \tan(\theta)} d\theta \\ & = 9 \int \sec^3(\theta) d\theta \\ & \therefore \text{ use reduction formula} \end{aligned}$$

## Example (x)

$$\int \frac{x^2}{x^2-9} dx$$

$$x = 3 \sec(\theta)$$
$$dx = 3 \sec(\theta) \tan(\theta) d\theta$$
$$\sqrt{x^2-9} = 3 \tan(\theta)$$

$$= \int \frac{(3 \sec(\theta))^2 (3 \sec(\theta) \tan(\theta))}{(3 \tan(\theta))^2} d\theta$$

$$= 3 \int \frac{\sec^3(\theta)}{\tan(\theta)} d\theta$$

~~$$= 3 \int \frac{1}{\cos^3(\theta)} \div \frac{\sin(\theta)}{\cos(\theta)} d\theta$$~~

~~$$= 3 \int \frac{1}{\sin(\theta) \cos^2(\theta)} d\theta$$~~

$$= 3 \int \frac{(1 + \tan^2(\theta)) \sec(\theta)}{\tan(\theta)} d\theta$$

$$= 3 \int (\csc(\theta) + \sec(\theta) \tan(\theta)) d\theta$$

$$= 3 (\ln |\csc(\theta) - \cot(\theta)| + \sec(\theta)) + C$$

$$= 3 \left( \ln \left| \frac{x}{\sqrt{x^2-9}} - \frac{3}{\sqrt{x^2-9}} \right| + \frac{x}{3} \right) + C$$

We will have another way of doing this example in the next section.

Note that these substitutions are all based on  $\cos^2(\theta) + \sin^2(\theta) = 1$ . We could also use hyperbolic functions b/c  $\cosh^2(x) - \sinh^2(x) = 1$