

§7.4 Integration of Rational Functions by Partial Fraction Decomposition

Note:

$$\frac{1}{x-1} + \frac{1}{x+2} + \frac{1}{x^2+1} \quad *$$

We can get a common denominator and write as a single fraction

$$= \frac{2x^3 + 2x^2 + 3x - 1}{(x-1)(x+2)(x^2+1)} \quad \text{X}$$

(I used the Together command in Mathematica to do this.)

What we will do in this section is start with a rational function, like ~~X~~, and "decompose" it into its parts, like *.

Example (i)

$$\int \frac{4x}{(x-1)(x+2)} dx$$

Linear factors of multiplicity one in the denominator:

$$\frac{4x}{(x-1)(x+2)} \stackrel{\text{set}}{=} \frac{A}{x-1} + \frac{B}{x+2}$$

where A and B are unknown constants

$$\Leftrightarrow 4x = A(x+2) + B(x-1)$$

Now choose x values to get 2 equations
in A and B .

$$x=1 : 4 = 3A$$

$$x=-2 : -8 = -3B$$

$$\text{So } A = \frac{4}{3} \text{ and } B = \frac{8}{3}$$

then

$$\begin{aligned} \int \frac{4x}{(x-1)(x+2)} dx &= \frac{4}{3} \int \frac{1}{x-1} dx + \frac{8}{3} \int \frac{1}{x+2} dx \\ &= \frac{4}{3} \ln|x-1| + \frac{8}{3} \ln|x+2| + C \end{aligned}$$

Example (ii)

$$\int \frac{5x^2 - 11x + 15}{x^2(x-3)} dx$$

Factor the denominator:

$$\frac{5x^2 - 11x + 15}{x^2(x-3)}$$

One factor of multiplicity 2, so

$$\frac{5x^2 - 11x + 15}{x^2(x-3)} \stackrel{\text{set}}{=} \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-3}$$

$$\Leftrightarrow 5x^2 - 11x + 15 = Ax(x-3) + B(x-3) + Cx^2$$

$$x=0 : 15 = -3B$$

$$x=3 : 45 - 33 + 15 = 9C$$

$$x=1 : 5 - 11 + 15 = -2A - 2B + C$$

$$\Rightarrow B = -5, C = 3, 9 = -2A + 10 + 3 \Rightarrow A = 2$$

So the integral becomes

$$\begin{aligned} & 2 \int \frac{1}{x} dx + 5 \int \frac{1}{x^2} dx + 3 \int \frac{1}{x-3} dx \\ &= 2 \ln|x| + \frac{5}{x} + 3 \ln|x-3| + C \\ &= \frac{5}{x} + \ln|x^2(x-3)^3| + C \end{aligned}$$

Example (iii)

$$\int \frac{3x^5 - 3x^3 + 25x^2 - 44x + 76}{x^4 - 16} dx$$

Here the integrand is not a "proper" fraction (5^{th} degree \div 4^{th} degree polynomials). So we do long division first:

$$\begin{array}{r} 3x \\ x^4 - 16 \sqrt{3x^5 - 0x^4 - 3x^3 + 25x^2 - 44x + 76} \\ \underline{- (3x^4)} \\ \hline -3x^3 + 25x^2 + 4x + 76 \end{array}$$

So the integral becomes

$$\int \left(3x + \frac{-3x^3 + 25x^2 + 4x + 76}{(x-2)(x+2)(x^2+4)} \right) dx$$

Now the decomposition of the fraction:

$$\frac{-3x^3 + 25x^2 + 4x + 76}{(x-2)(x+2)(x^2+4)} = \frac{A}{(x-2)} + \frac{B}{(x+2)} + \frac{Cx+D}{(x^2+4)}$$

$$\Leftrightarrow -3x^3 + 25x^2 + 4x + 76 = A(x+2)(x^2+4) + B(x-2)(x^2+4)$$

$$+ (Cx+D)(x-2)(x+2)$$

We could choose 4 values of x as before, but this time we will equate coefficients of the polynomials on either side of the equation.

$$\begin{aligned} x^3 &: -3 = A + B + C \\ x^2 &: 25 = 2A - 2B + D \\ x^1 &: 4 = 4A + 4B - 4C \\ x^0 &: 76 = 8A - 8B - 4D \end{aligned}$$

In a TI calculator under Matrix

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 0 & -3 \\ 2 & -2 & 0 & 1 & 25 \\ 4 & 4 & -4 & 0 & 4 \\ 8 & -8 & 0 & -4 & 76 \end{array} \right) \xrightarrow{\text{ref}} \left(\begin{array}{ccccc} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & -6 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{array} \right)$$

This means that the solution to the system of linear equations in A, B, C and D is

$A=5, B=-6, C=-2, D=3$
and the integral becomes

$$\begin{aligned} & \int 3x \, dx + 5 \int \frac{1}{x-2} \, dx - 6 \int \frac{1}{x+2} \, dx - 2 \int \frac{x}{x^2+4} \, dx + 3 \int \frac{1}{x^2+4} \, dx \\ &= \frac{3}{2}x^2 + 5 \ln|x-2| - 6 \ln|x+2| - \ln(x^2+4) + \frac{3}{2} \tan^{-1}\left(\frac{x}{2}\right) + C \end{aligned}$$

Example (iv)

$$\int \frac{1}{2\sqrt{x+3} + x} dx$$

On this one we do a "rationalizing" substitution.

Let $u = \sqrt{x+3} \Leftrightarrow x = u^2 - 3, dx = 2u du$

$$= \int \frac{2u}{2u + u^2 - 3} du$$

$$\frac{2u}{(u+3)(u-1)} \stackrel{\text{set}}{=} \frac{A}{u+3} + \frac{B}{u-1}$$

$$\Leftrightarrow 2u = A(u-1) + B(u+3)$$

$$\begin{aligned} u=1 : \quad 2 &= 4B \\ u=-3 : \quad -6 &= -4A \end{aligned} \Rightarrow A = \frac{3}{2}, B = \frac{1}{2}$$

So the integral is

$$\frac{3}{2} \int \frac{1}{u+3} du + \frac{1}{2} \int \frac{1}{u-1} du$$

$$= \frac{3}{2} \ln|u+3| + \frac{1}{2} \ln|u-1| + C$$

$$= \frac{3}{2} \ln|\sqrt{x+3} + 3| + \frac{1}{2} \ln|\sqrt{x+3} - 1| + C$$

Example (v) Write the follow fraction in decomposed form, but do not solve for the unknown constants.

$$\frac{1}{x^2(x+1)^3(x^2+1)^2}$$

Here we have: x is a linear factor of multiplicity 2, $x+1$ is a linear factor of multiplicity 3 and x^2+1 is a quadratic factor of multiplicity 2. So

$$\begin{aligned}\frac{1}{x^2(x+1)^3(x^2+1)^2} &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)^3} \\ &\quad + \frac{Fx+G}{x^2+1} + \frac{Hx+I}{(x^2+1)^2}\end{aligned}$$

Note: All polynomials with real coefficients can be factored into linear and/or quadratic factors (possibly with multiplicities greater than one).