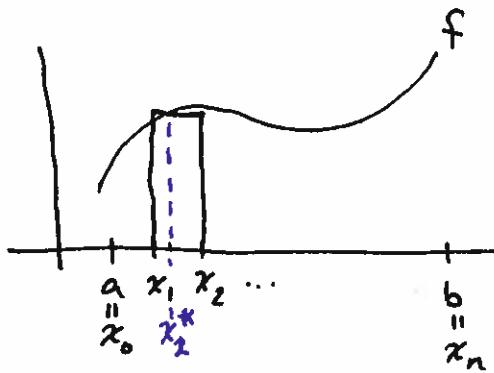


§ 7.7 Approximate Integration

Recall:



$$\Delta x = \frac{b-a}{n}, \quad x_k = a + k\Delta x, \quad x_k^* \in [x_{k-1}, x_k]$$

$$\int_a^b f(x) dx \approx \sum_{k=1}^n f(x_k^*) \Delta x$$

These are rectangular approximations to an integral. If $x_k^* = x_k$, then it is the right-hand method, R_n . If $x_k^* = x_{k-1}$, then it is the left-hand method, L_n .

Midpoint Rule (M_n)

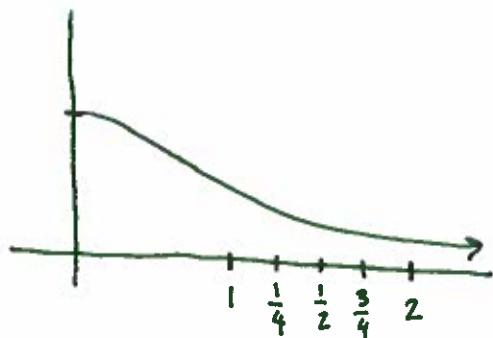
$x_k^* = a + (k - \frac{1}{2})\Delta x$ or the midpoint of the interval. The midpoint rule has an error of

$$|E_{M_n}| \leq \frac{K(b-a)^3}{24 n^2}$$

where $|f''(x)| \leq K$ for $a \leq x \leq b$.

Example (i)

Use the midpoint rule with $n=4$ to approximate $\int_1^2 e^{-x^2} dx$.



$$\Delta x = \frac{2-1}{4} = \frac{1}{4}$$

$$x_1^* = \frac{1}{8}, x_2^* = \frac{3}{8}, x_3^* = \frac{5}{8}$$

$$x_4^* = \frac{7}{8}$$

So if $f(x) = e^{-x^2}$, then

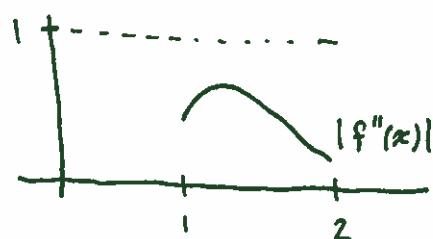
$$\begin{aligned} M_4 &= (f(1.125) + f(1.375) + f(1.625) + f(1.875))/4 \\ &\approx 0.13352 \end{aligned}$$

Note $\int_1^2 e^{-x^2} dx \approx 0.135257$

Example (ii)

What is the theoretical bound on the error in example (i)?

Using Mathematica:



We could use $K=1$ (or .91) to get

$$|E_{M_4}| \leq \frac{1(2-1)^3}{(24)(4^2)} \approx .0026$$

Note that the actual error is about $|0.135257 - 0.13352| = 0.001737$. The actual error should be \leq theoretical error.

Trapezoidal Rule

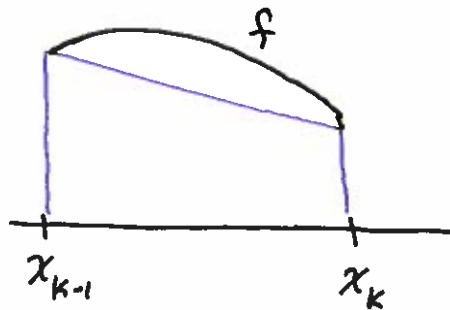
$$T_n = \frac{\Delta x}{2} (f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n))$$

with error

$$|E_{T_n}| \leq \frac{k(b-a)^3}{12 n^2}$$

where $|f''(x)| \leq k$ for $a \leq x \leq b$.

This comes from looking at trapezoids instead of rectangles:



$$\int_{x_{k-1}}^{x_k} f(x) dx \approx \frac{\Delta x}{2} (f(x_{k-1}) + f(x_k))$$

Note: for the Midpoint Rule :

$$\int_{x_{k-1}}^{x_k} f(x) dx = f\left(\frac{x_{k-1} + x_k}{2}\right) \Delta x$$

Also note that the error bound for Midpoint Rule is a little bit better than the one for the trapezoidal Rule.

Example (iii)

Approximate $\int_1^2 e^{-x^2} dx$ by using T_4 .

See example (i). $\Delta x = \frac{1}{4}$, $x_0 = 1$, $x_1 = 1.25$,
 $x_2 = 1.5$, $x_3 = 1.75$, $x_4 = 2$

$$\begin{aligned} T_4 &= \frac{\Delta x}{2} (f(x_0) + 2f(x_1) + \dots + f(x_4)) \\ &= \frac{1}{8} (f(1) + 2f(1.25) + 2f(1.5) + 2f(1.75) + f(2)) \\ &\approx 0.13872 \end{aligned}$$

Note: not quite as good as M_4

So why do we look at the Trapezoidal Rule?
 Look at the following example:

Example (iv)

Approximate $\int_1^2 e^{-x^2} dx$ by using T_8 . Try to use T_4 in this calculation.

Now $\Delta x = \frac{2-1}{8} = \frac{1}{8}$ and $x_0 = 1$, $x_1 = 1.125$, $x_2 = 1.25$,
 $x_3 = 1.375$, $x_4 = 1.5$, $x_5 = 1.625$, $x_6 = 1.75$, $x_7 = 1.875$,
 $x_8 = 2$. Note that x_0, x_2, x_4, x_6, x_8 are the "nodes"
 that we used in T_4 . So

$$\begin{aligned} T_8 &= \frac{1}{2} T_4 + \frac{\Delta x}{2} (2f(x_1) + 2f(x_3) + 2f(x_5) + 2f(x_7)) \\ &= \frac{1}{2} T_4 + \Delta x (f(x_1) + f(x_3) + f(x_5) + f(x_7)) \\ &\approx 0.13612 \end{aligned}$$

Interval doubling

One method of approximating an integral is to use a Rule over and over again for $n=1, 2, 4, 8, \dots$. We stop when two successive approximations are "close" to each other.

For Midpoint Rule, when we double the n , we need to calculate the function at completely new nodes. We can not reuse any information from the ~~the~~ previous approximation.

For the Trapezoidal Rule, when we interval double, we can use the previous approximation. Thus, at the step where $n=2^j$, we only need to evaluate the function at 2^{j-1} nodes.

Simpson's Rule

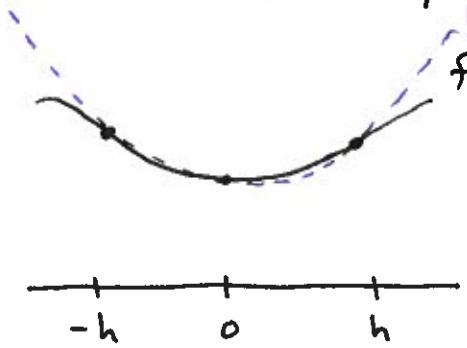
$$S_n = \frac{\Delta x}{3} \left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right)$$

With error bounds

$$|E_{S_n}| \leq \frac{M(b-a)^5}{180 n^4}$$

Where $|f^{(4)}(x)| \leq M$ for $a \leq x \leq b$.

This comes from looking at quadratic curves:



$$f(x) \approx y(x) = ax^2 + bx + c$$

$$\text{where } f(-h) = y(-h) = ah^2 - bh + c$$

$$f(0) = y(0) = c$$

$$f(h) = y(h) = ah^2 + bh + c$$

$$\begin{aligned} \int_{-h}^h f(x) dx &\approx \int_{-h}^h y(x) dx = \frac{1}{3}ax^3 + \frac{1}{2}bx^2 + cx \Big|_{-h}^h \\ &= \frac{2}{3}ah^3 + 2ch \\ &= \frac{h}{3}(2ah^2 + 6c) \end{aligned}$$

$$\text{But } f(-h) + f(h) = 2ah^2 + 2c \text{ and } f(0) = c, \text{ so}$$

$$\int_{-h}^h y(x) dx = \frac{h}{3} (f(-h) + 4f(0) + f(h))$$

Thus

$$\int_{x_k}^{x_{k+2}} f(x) dx \approx \frac{\Delta x}{3} (f(x_k) + 4f(x_{k+1}) + f(x_{k+2}))$$

Note: for Simpson's Rule n MUST be even.

The application of Simpson's Rule is similar to the examples above. Now an example using the error formulas

Example (v)

On the integral $\int_1^2 e^{-x^2} dx$, Find the n's so that the errors in the Midpoint Rule and Simpson's Rule are less than 10^{-6} .

Using Mathematica to graph $|f''(x)|$ and $|f^{(4)}(x)|$

we have $K=1$ and $M=8$, so

$$|E_{M_n}| \leq \frac{1}{24n^2} \stackrel{\text{set}}{\leq} 10^{-6}$$

$$\Rightarrow 24n^2 \geq 10^6$$

$$n \geq \sqrt{\frac{10^6}{24}} = 204.124$$

so M_{205} works

$$|E_{S_n}| \leq \frac{8}{180n^4} \stackrel{\text{set}}{\leq} 10^{-6}$$

$$\Rightarrow \frac{180n^4}{8} \geq 10^6$$

$$n \geq \sqrt[4]{\frac{8}{180} \times 10^6} \approx 14.5196$$

so S_{16} work (why $n=16$ and not 15?)