

## §7.8 Improper Integrals

The working definition for definite integrals is

Let  $f$  be continuous on  $[a, b]$ , then

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k^*) \Delta x$$

where  $\Delta x = \frac{b-a}{n}$  and  $a + (k-1)\Delta x \leq x_k^* \leq a + k\Delta x$ .

While this definition is perfectly fine for most situations, we need to extend it to cover cases where  $f$  is not continuous at a finite number of points in  $[a, b]$ .

Example (i)  $\int_0^1 \frac{1}{\sqrt{x}} dx$

Here, the integrand ( $f(x) = x^{-1/2}$ ) is not continuous, or defined, at  $x=0$ .

Def  $\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx$

provided  $f$  is continuous on  $(a, b]$  and the limit exists.

For this example

$$\int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{t \rightarrow 0^+} \int_t^1 x^{-1/2} dx = \lim_{t \rightarrow 0^+} 2(\sqrt{1} - \sqrt{t}) = 2$$

A similar process can be used if  $f$  has a discontinuity at  $b$ .

Example (ii) for what values of  $p$  is

$$\int_0^1 \frac{1}{x^p} dx$$

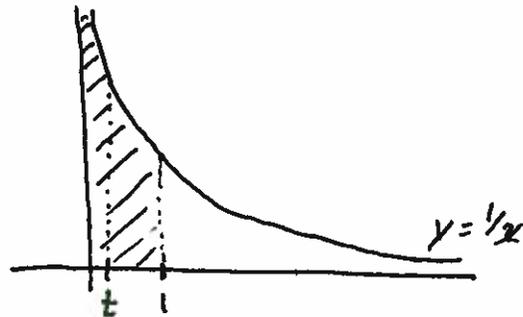
defined?

$$\begin{aligned} \int_0^1 \frac{1}{x^p} dx &= \lim_{t \rightarrow 0^+} \int_t^1 x^{-p} dx \\ &= \lim_{t \rightarrow 0^+} \left. \frac{x^{1-p}}{1-p} \right|_t^1, \text{ provided } p \neq 1 \\ &= \lim_{t \rightarrow 0^+} \frac{1 - t^{1-p}}{1-p} \\ &= \begin{cases} \frac{1}{1-p}, & p < 1 \\ \text{DNE}, & p > 1 \end{cases} \end{aligned}$$

Note 
$$\begin{aligned} \int_0^1 \frac{1}{x} dx &= \lim_{t \rightarrow 0^+} \int_t^1 \frac{1}{x} dx \\ &= \lim_{t \rightarrow 0^+} \left. \ln|x| \right|_t^1 \\ &= \lim_{t \rightarrow 0^+} -\ln(t) = \infty \end{aligned}$$

So 
$$\int_0^1 \frac{1}{x^p} dx = \begin{cases} \frac{1}{1-p}, & p < 1 \\ \text{DNE}, & p \geq 1 \end{cases}$$

If an improper integral has a value we say the integral converges. Otherwise, we say it diverges.



For  $y = \frac{1}{x}$ , the "area under the curve" on  $(0, 1]$  diverges to  $\infty$ .

Example (iii)

$$\int_0^{\pi/2} \tan(x) dx$$

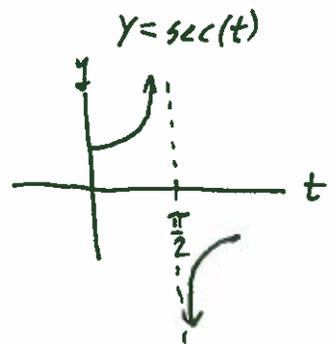
$$= \lim_{t \rightarrow \frac{\pi}{2}^-} \int_0^t \tan(x) dx$$

$$= \lim_{t \rightarrow \frac{\pi}{2}^-} \ln|\sec(x)| \Big|_0^t$$

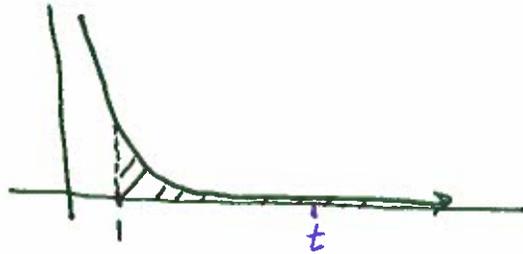
$$= \lim_{t \rightarrow \frac{\pi}{2}^-} \ln|\sec(t)| - \ln(1)$$

$$= \infty$$

So the integral diverges.



Example (iv)  $\int_1^{\infty} \frac{1}{x^2} dx$



Def  $\int_c^{\infty} f(x) dx = \lim_{t \rightarrow \infty} \int_c^t f(x) dx$

provided  $f$  is continuous for  $x \geq c$  and the limit exists.

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^2} dx &= \lim_{t \rightarrow \infty} \int_1^t x^{-2} dx \\ &= \lim_{t \rightarrow \infty} \left. \frac{x^{-1}}{-1} \right|_1^t \\ &= \lim_{t \rightarrow \infty} \left( 1 - \frac{1}{t} \right) = 1 \end{aligned}$$

So the integral converges to a value of 1.

$$\int_1^{\infty} \frac{1}{x^p} dx = \begin{cases} \text{converges, } p > 1 \\ \text{diverges, } p \leq 1 \end{cases}$$

We call this the  $p$ -test for integrals.

Comparison test Let  $f$  and  $g$  be decreasing functions and suppose  $0 \leq f(x) \leq g(x)$  for each  $x \in I$  where  $I$  is an interval (possibly infinite), then

(a)  $\int_I f(x) dx$  diverges to  $\infty \Rightarrow \int_I g(x) dx$  diverges also.

(b)  $\int_I g(x) dx$  converges  $\Rightarrow \int_I f(x) dx$  converges.

Example (v) Show that  $\int_0^{\infty} \frac{1}{x^3+1} dx$  converges.

$$\text{First, } \int_0^{\infty} \frac{1}{x^3+1} dx = \underbrace{\int_0^1 \frac{1}{x^3+1} dx}_{I_1} + \underbrace{\int_1^{\infty} \frac{1}{x^3+1} dx}_{I_2}$$

$I_1$  is well defined (not improper).

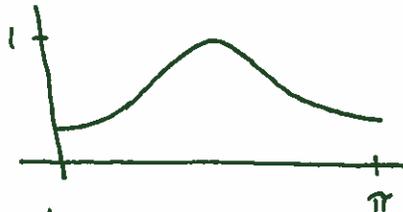
~~On~~  $I_2$ : on  $[1, \infty)$ ,  $\frac{1}{x^3+1} < \frac{1}{x^3}$  and  $\int_1^{\infty} \frac{1}{x^3} dx$

converges by the p-test for integrals. Therefore,  $I_2$  converges by the comparison test.

Thus  $\int_0^{\infty} \frac{1}{x^3+1} dx$  is the sum of two convergent integrals and converges.

Example (vi)  $\int_0^{\pi} \frac{1}{9\cos^2(x) + \sin^2(x)} dx$

graph the integrand in ~~the~~ Mathematica:



To do this integral we note that

$$\int_0^{\pi} \frac{1}{9\cos^2(x) + \sin^2(x)} dx = \int_0^{\pi} \frac{\sec^2(x)}{9 + \tan^2(x)} dx$$

Now let  $u = \tan(x)$ ,  $du = \sec^2(x) dx \Rightarrow$

$$\int_{\tan(0)}^{\tan(\pi)} \frac{1}{9 + u^2} du = 0 \quad \text{b/c } \tan(0) = \tan(\pi) = 0$$

Is the area under the curve zero?

No! What went wrong is that the substitution  $u = \tan(x)$  is "improper".  $\tan(x)$  has a discontinuity at  $x = \frac{\pi}{2}$ . So we need to split the integral at  $\frac{\pi}{2}$ . Or notice the symmetry:

$$\int_0^{\pi} \frac{\sec^2(x)}{9 + \tan^2(x)} dx = 2 \int_0^{\pi/2} \frac{\sec^2(x)}{9 + \tan^2(x)} dx$$

Now we can do the substitution to get

$$2 \int_0^{\infty} \frac{1}{9 + u^2} du = 2 \lim_{t \rightarrow \infty} \frac{1}{3} \tan^{-1}\left(\frac{t}{3}\right) = \frac{2}{3} \left(\frac{\pi}{2}\right) = \frac{\pi}{3}$$

Example (vii) Consider the curve  $y = x^{-2/3}$  on the interval  $[1, \infty)$ . Find the area and the volume of rotation about the  $x$ -axis.

$$\text{Area} = \int_1^{\infty} \frac{1}{x^{2/3}} dx = \infty \text{ by } p\text{-test } (p = \frac{2}{3} \leq 1)$$

$$\text{Volume} = \pi \int_1^{\infty} \left(\frac{1}{x^{2/3}}\right)^2 dx \text{ by disk method}$$

$$= \pi \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^{4/3}} dx$$

$$= \pi \lim_{t \rightarrow \infty} \left. \frac{x^{-1/3}}{-1/3} \right|_1^t$$

$$= 3\pi \lim_{t \rightarrow \infty} \left(1 - \frac{1}{t^{1/3}}\right)$$

$$= 3\pi$$

Fun!!!

Final thoughts: the examples given all had an infinity somewhere in the problem. We handle that infinity by converting the integral into a limit of proper integrals. Jump discontinuities and "hole" in the graph can be handled in a similar way.